CSE4421/5324: Introduction to Robotics

Contact Information

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- EECS4421/5324
  lectures Monday, Wednesday, Friday 1:30-2:30PM (SLH C)
  Lab 1 Thursday 12:30-2:30, Prism 1004
  Lab 2 Thursday 2:30-4:30, Prism 1004
- www.eecs.yorku.ca/course/4421
  (web site not complete yet)

General Course Information

- introduces the basic concepts of robotic manipulators and autonomous systems. After a review of some fundamental mathematics the course examines the mechanics and dynamics of robot arms, mobile robots, their sensors and algorithms for controlling them.

Textbook

- no required textbook
- first 6 weeks of course uses notation consistent with *Robot Modeling and Control* by MW Spong, S Hutchinson, M Vidyasagar

Assessment

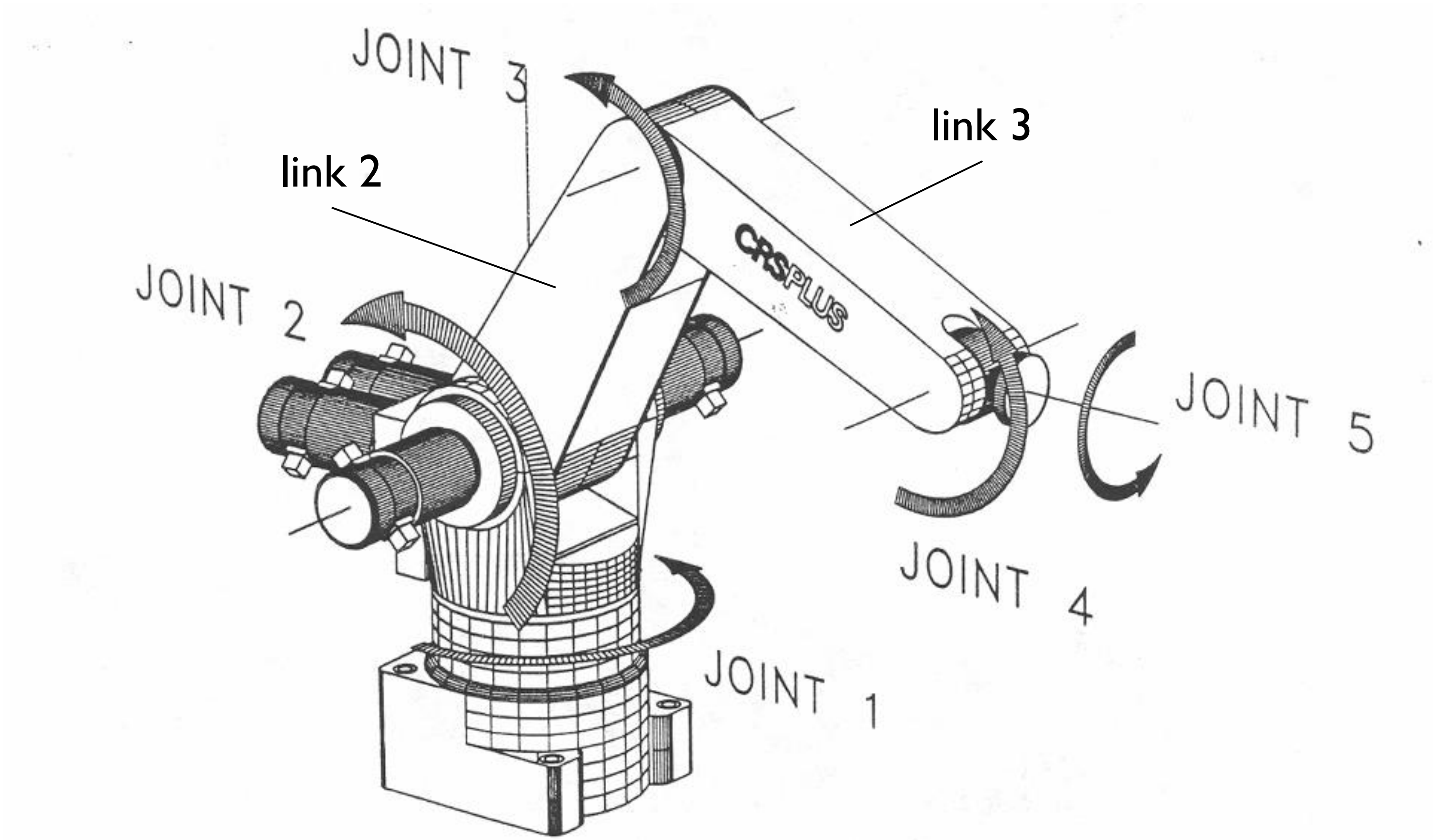
- labs/assignments 6 x 5%
- midterm, 30%
- exam, 40%

Introduction to manipulator kinematics

Robotic Manipulators

- a robotic manipulator is a kinematic chain
  - i.e. an assembly of pairs of rigid bodies that can move respect to one another via a mechanical constraint
- the rigid bodies are called *links*
- the mechanical constraints are called *joints*

A150 Robotic Arm



Joints

- most manipulator joints are one of two types

1. revolute (or rotary)
   - like a hinge

2. prismatic (or linear)
   - like a piston

- our convention: joint i connects link $i - 1$ to link i
  - when joint i is actuated, link i moves

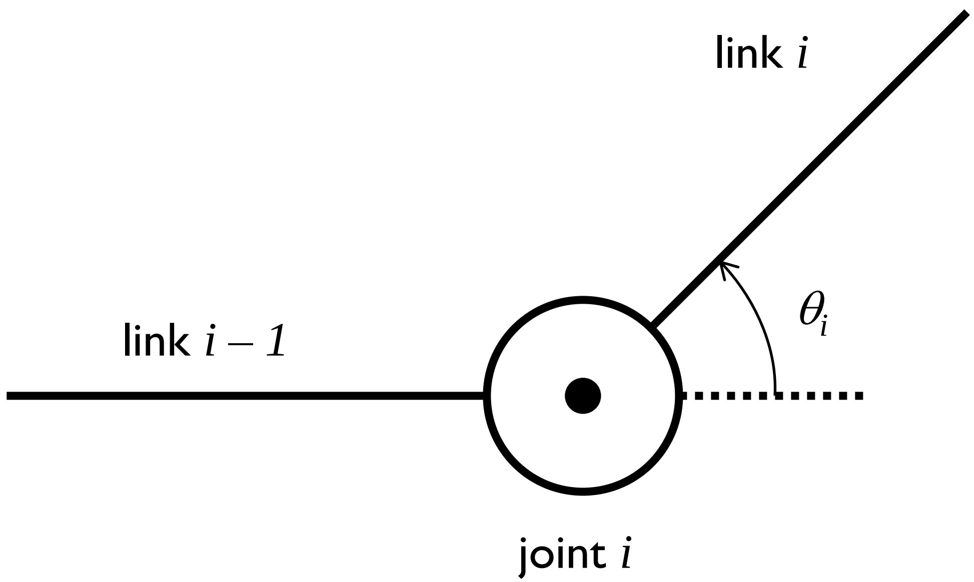
Joint Variables

- revolute and prismatic joints are one degree of freedom (DOF) joints; thus, they can be described using a single numeric value called a joint variable
- q_i : joint variable for joint i

1. revolute
   - $q_i = \theta_i$: angle of rotation of link i relative to link $i - 1$
2. prismatic
   - $q_i = d_i$: displacement of link i relative to link $i - 1$

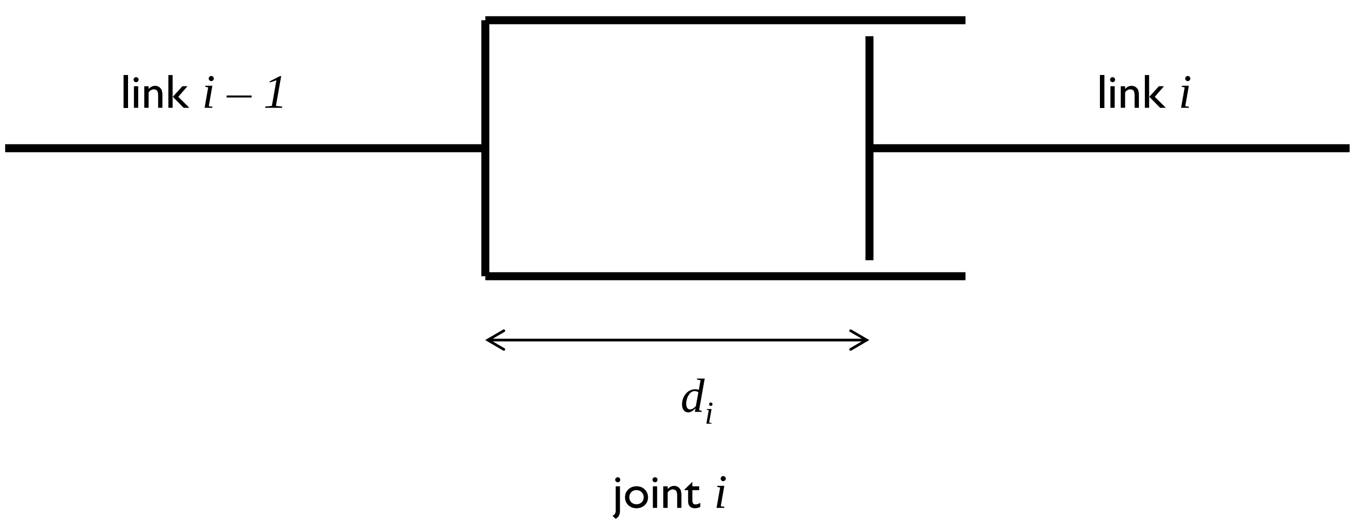
Revolute Joint Variable

- revolute
  - like a hinge
  - allows relative rotation about a fixed axis between two links
    - axis of rotation is the z axis by convention
- joint variable $q_i = \theta_i$: angle of rotation of link i relative to link $i - 1$



Prismatic Joint Variable

- prismatic
  - like a piston
  - allows relative translation along a fixed axis between two links
    - axis of translation is the z axis by convention
  - joint variable $q_i = d_i$: displacement of link i relative to link $i-1$

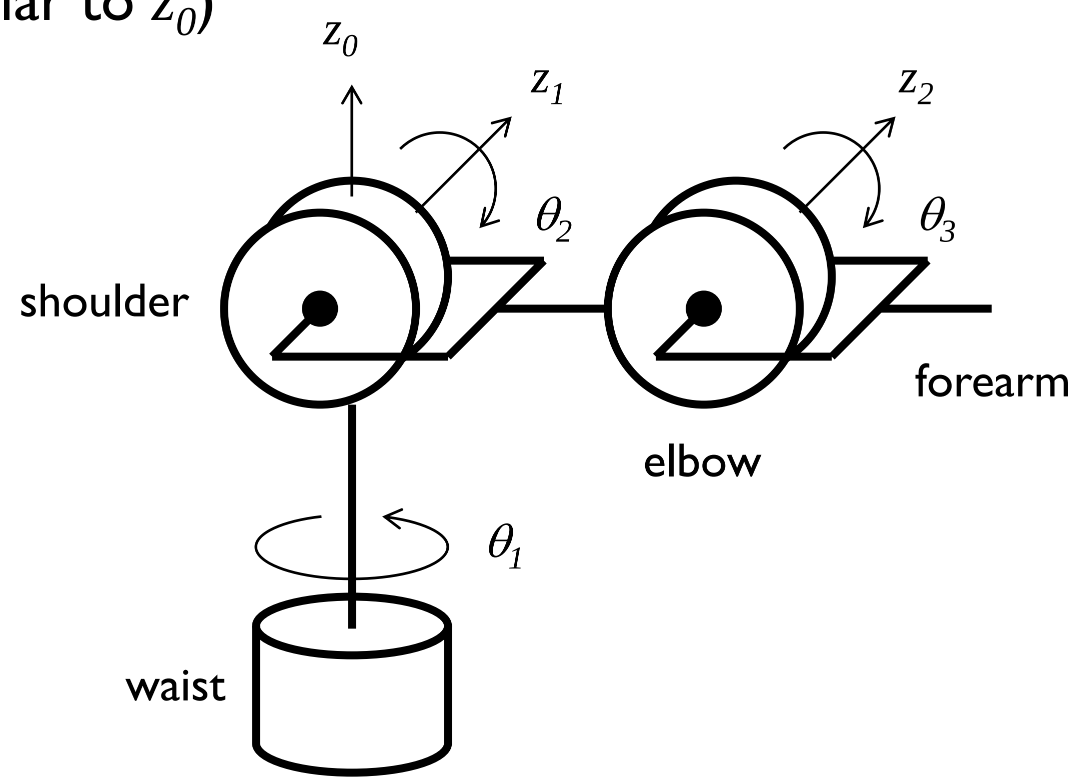


Common Manipulator Arrangments

- most industrial manipulators have six or fewer joints
  - the first three joints are the arm
  - the remaining joints are the wrist
- it is common to describe such manipulators using the joints of the arm
  - R: revolute joint
  - P: prismatic joint

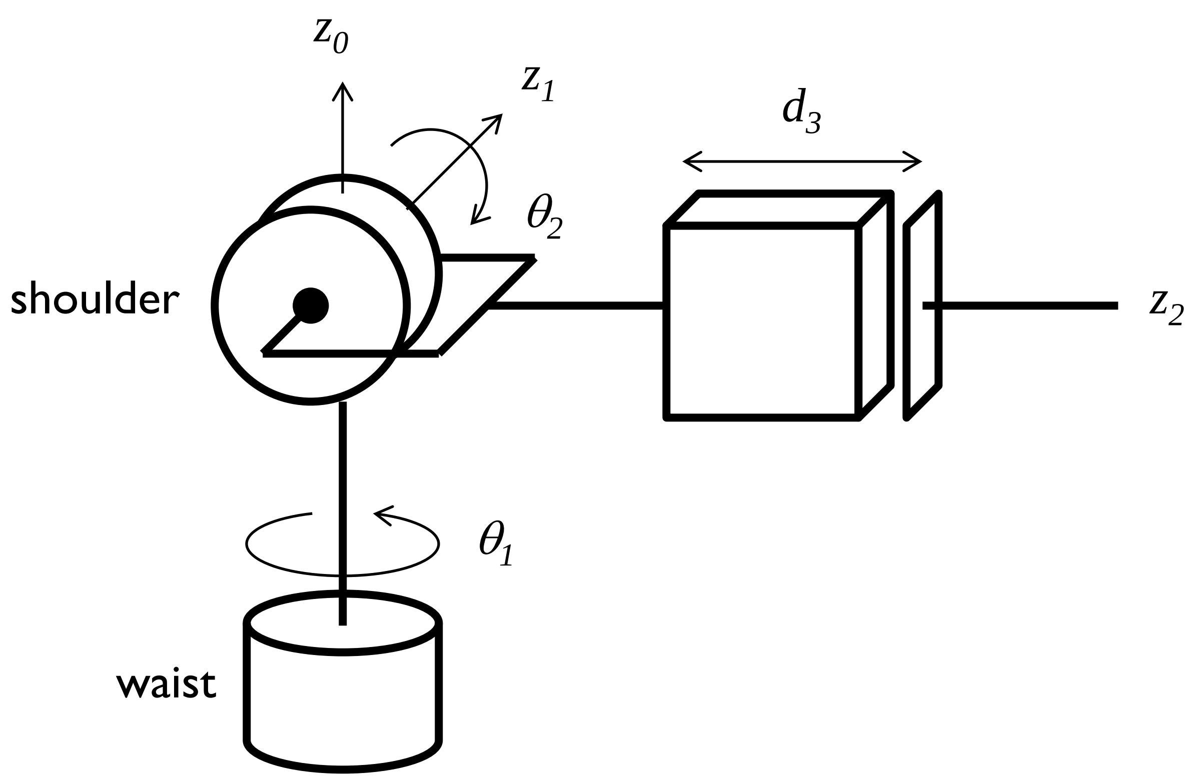
Articulated Manipulator

- RRR (first three joints are all revolute)
- joint axes
  - z_0 : waist
  - z_1 : shoulder (perpendicular to z_0)
  - z_2 : elbow (parallel to z_1)



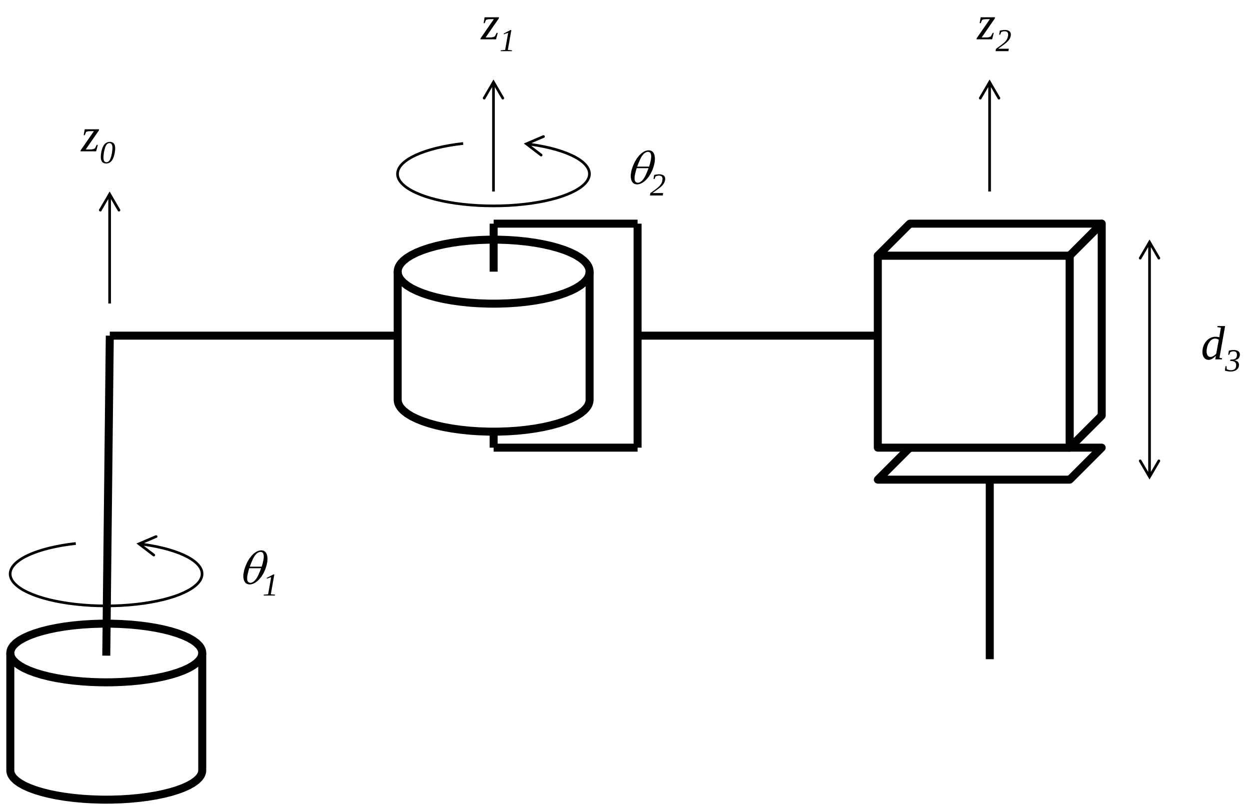
Spherical Manipulator

- RRP
- Stanford arm
  - http://infolab.stanford.edu/pub/voy/museum/pictures/display/robots/IMG_2404ArmFrontPeekingOut.JPG



SCARA Manipulator

- RRP
- Selective Compliant Articulated Robot for Assembly
  - http://www.robots.epson.com/products/g-series.htm

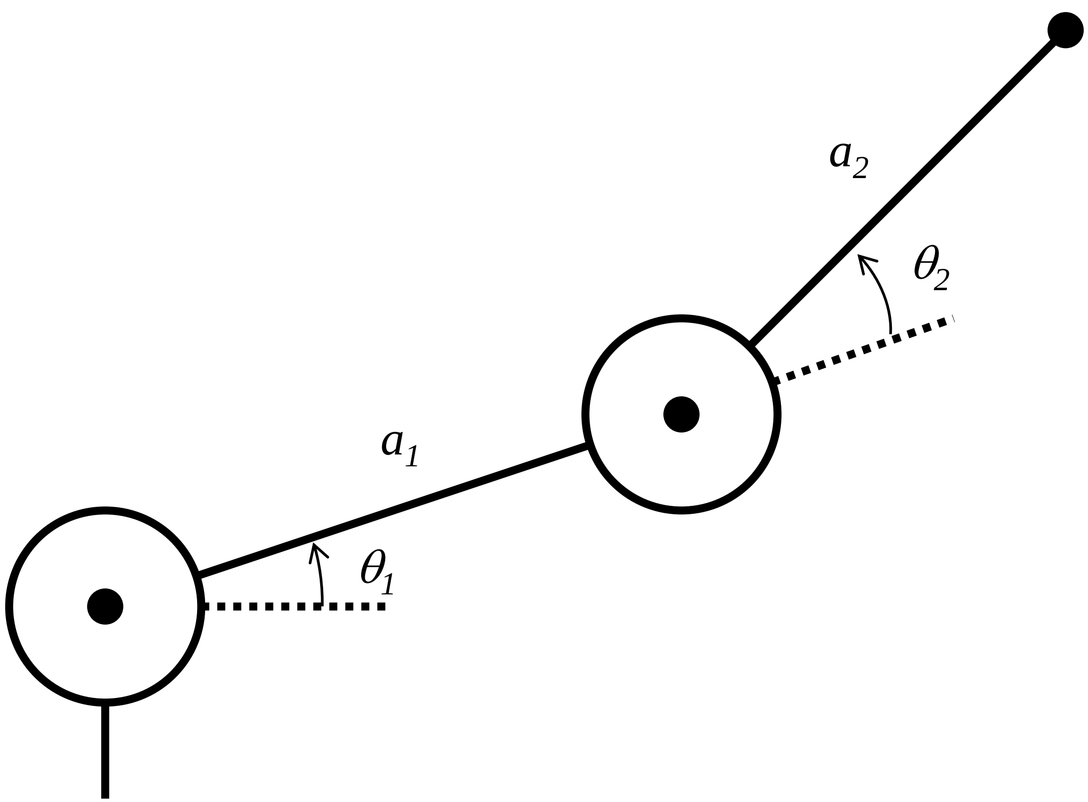


Parallel Robots

- all of the preceding examples are examples of serial chains
  - base (link 0) is connected to link 1 by a joint
  - link 1 is connected to link 2 by a joint
  - link 2 is connected to link 3 by a joint ... and so on
- a parallel robot is formed by connecting two or more serial chains
  - https://www.youtube.com/watch?v=p1Lrz0gPvOA

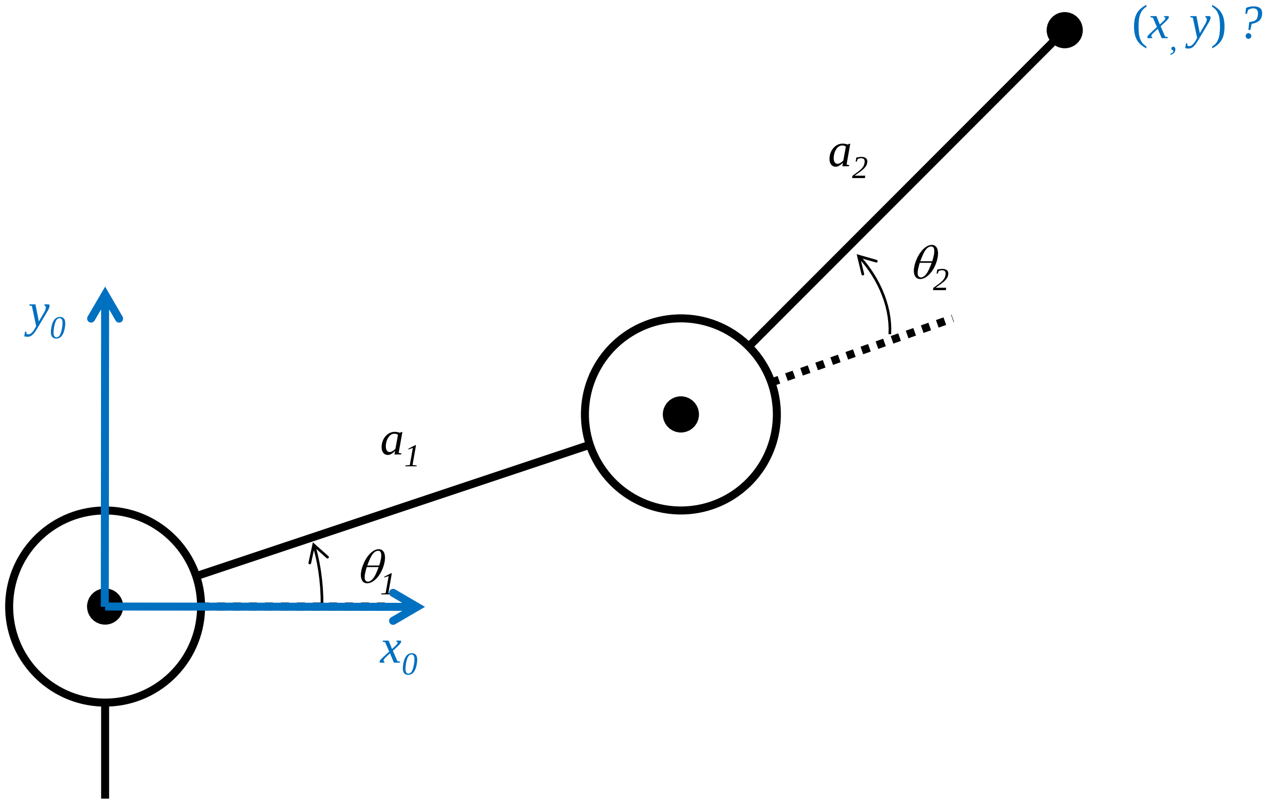
Forward Kinematics

- given the joint variables and dimensions of the links what is the position and orientation of the end effector?



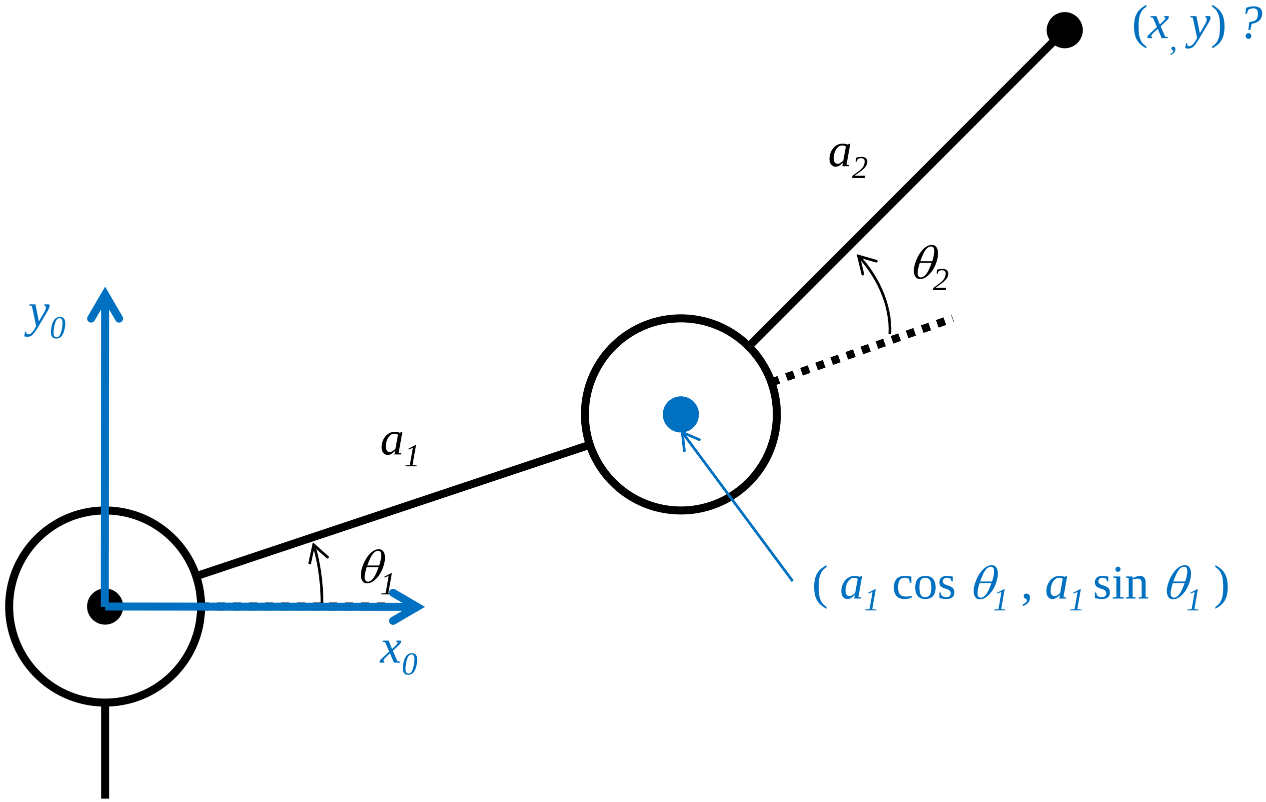
Forward Kinematics

- choose the base coordinate frame of the robot
  - we want (x, y) to be expressed in this frame



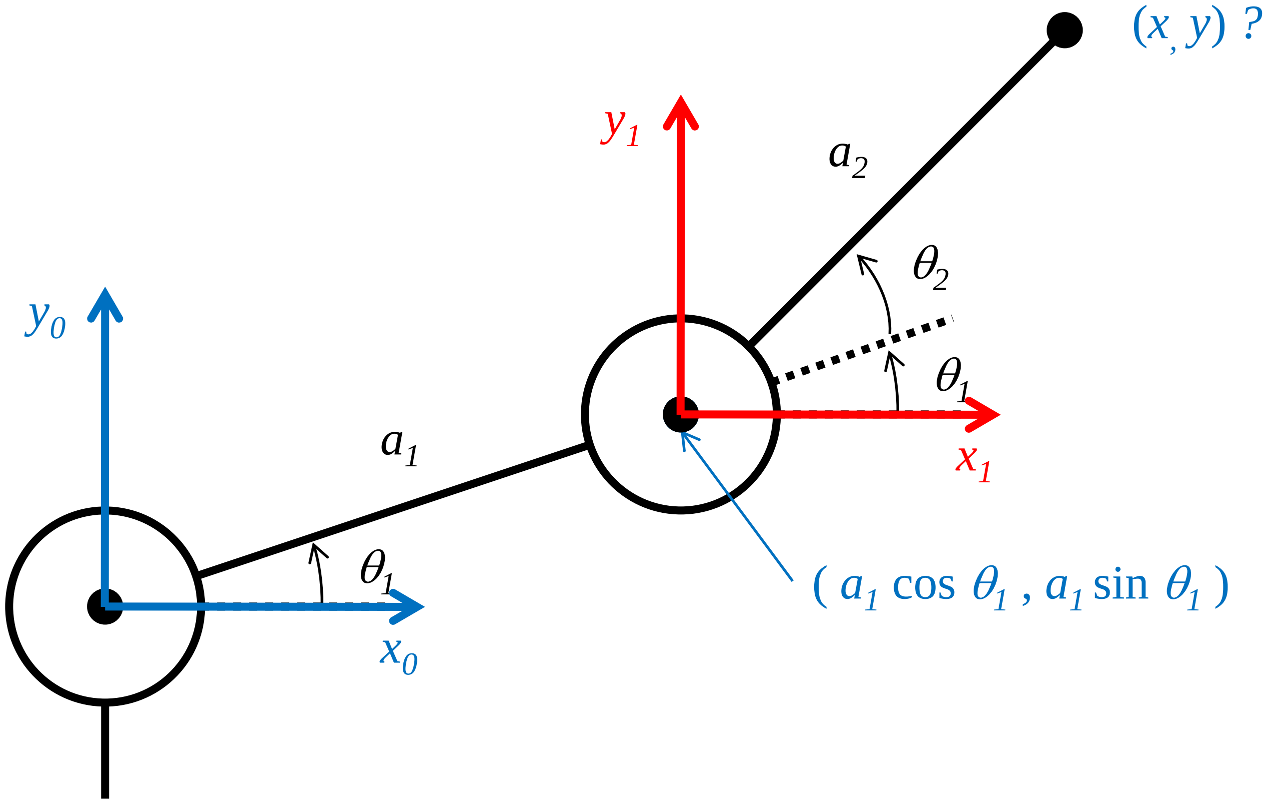
Forward Kinematics

- notice that link *1* moves in a circle centered on the base frame origin



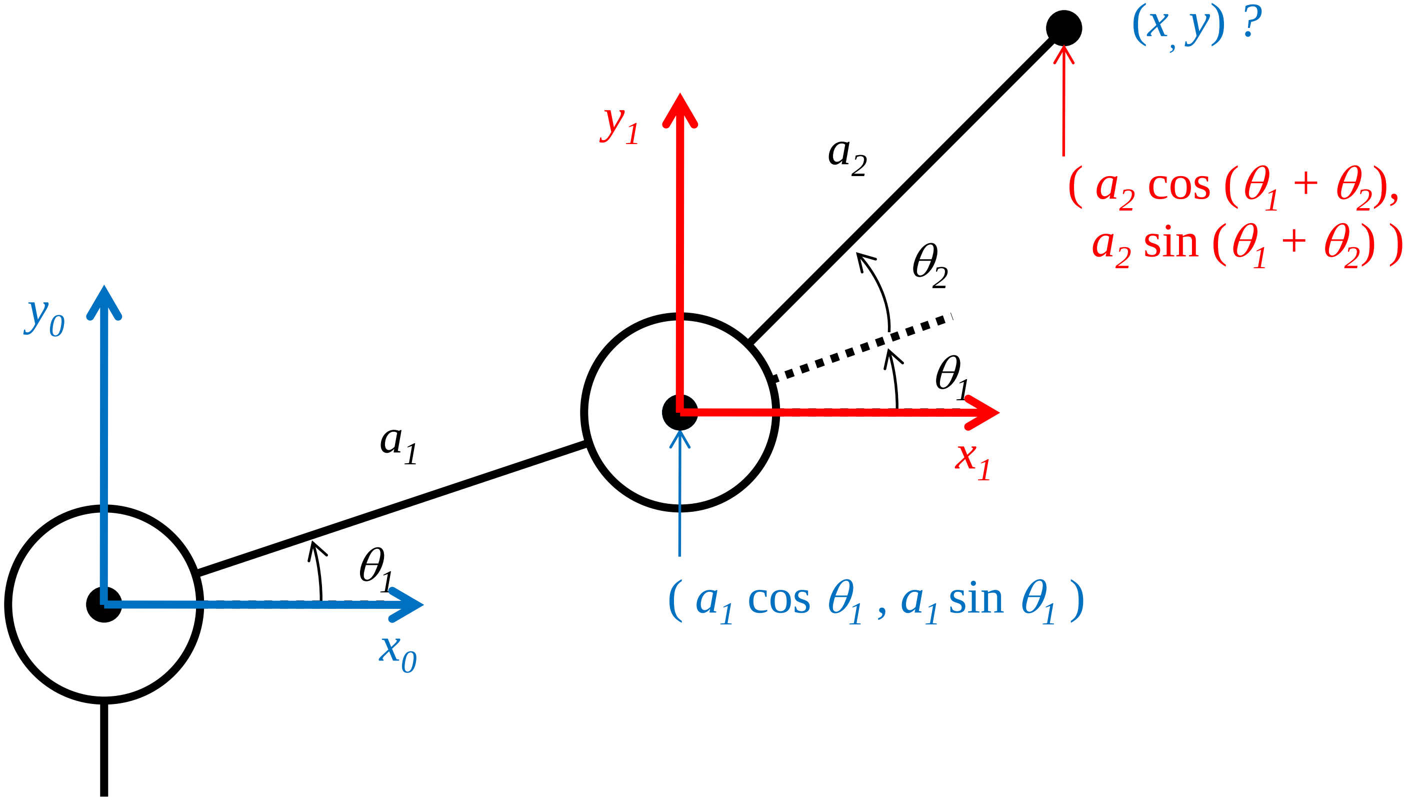
Forward Kinematics

- choose a coordinate frame with origin located on joint *2* with the same orientation as the base frame



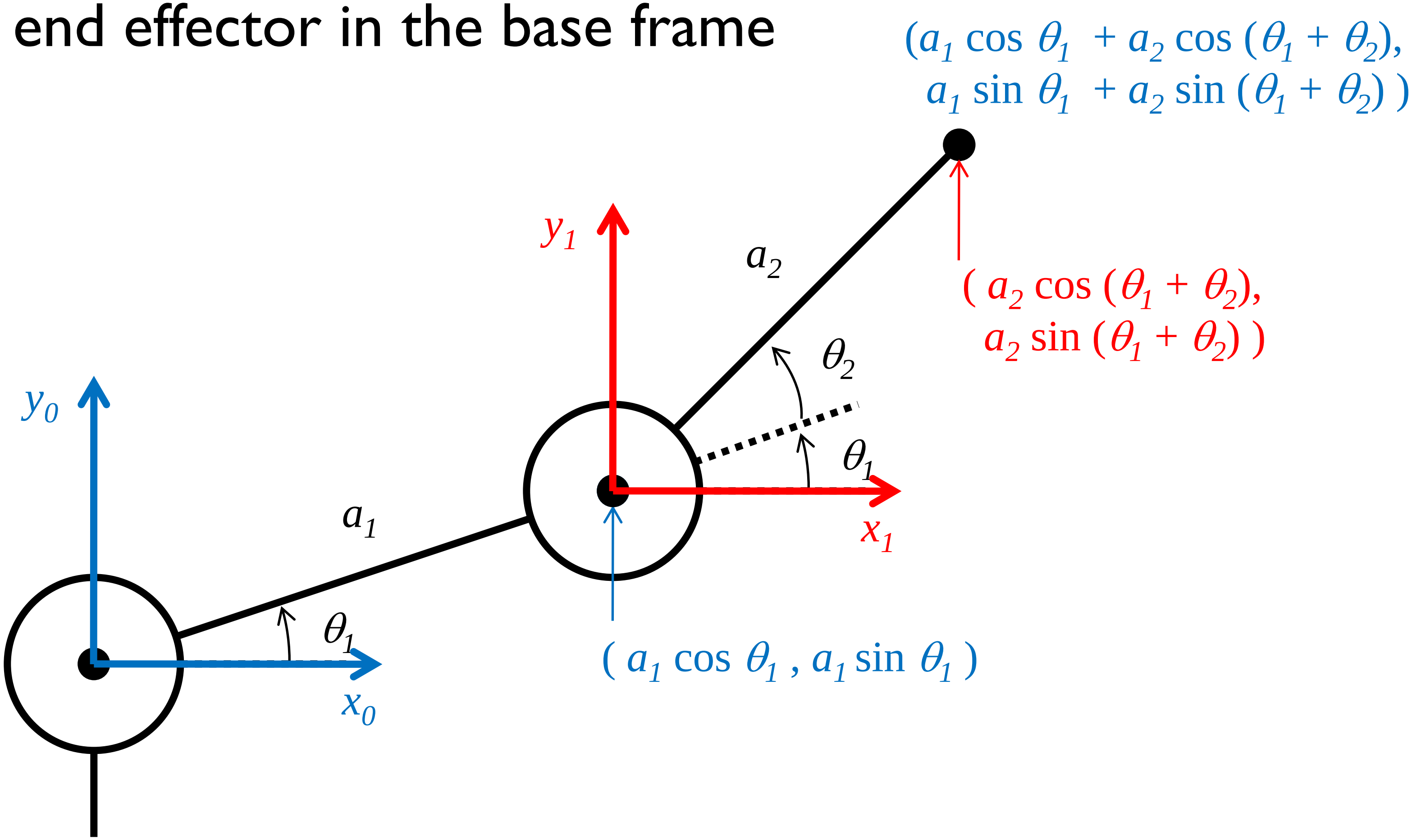
Forward Kinematics

- notice that link *2* moves in a circle centered on frame *1*



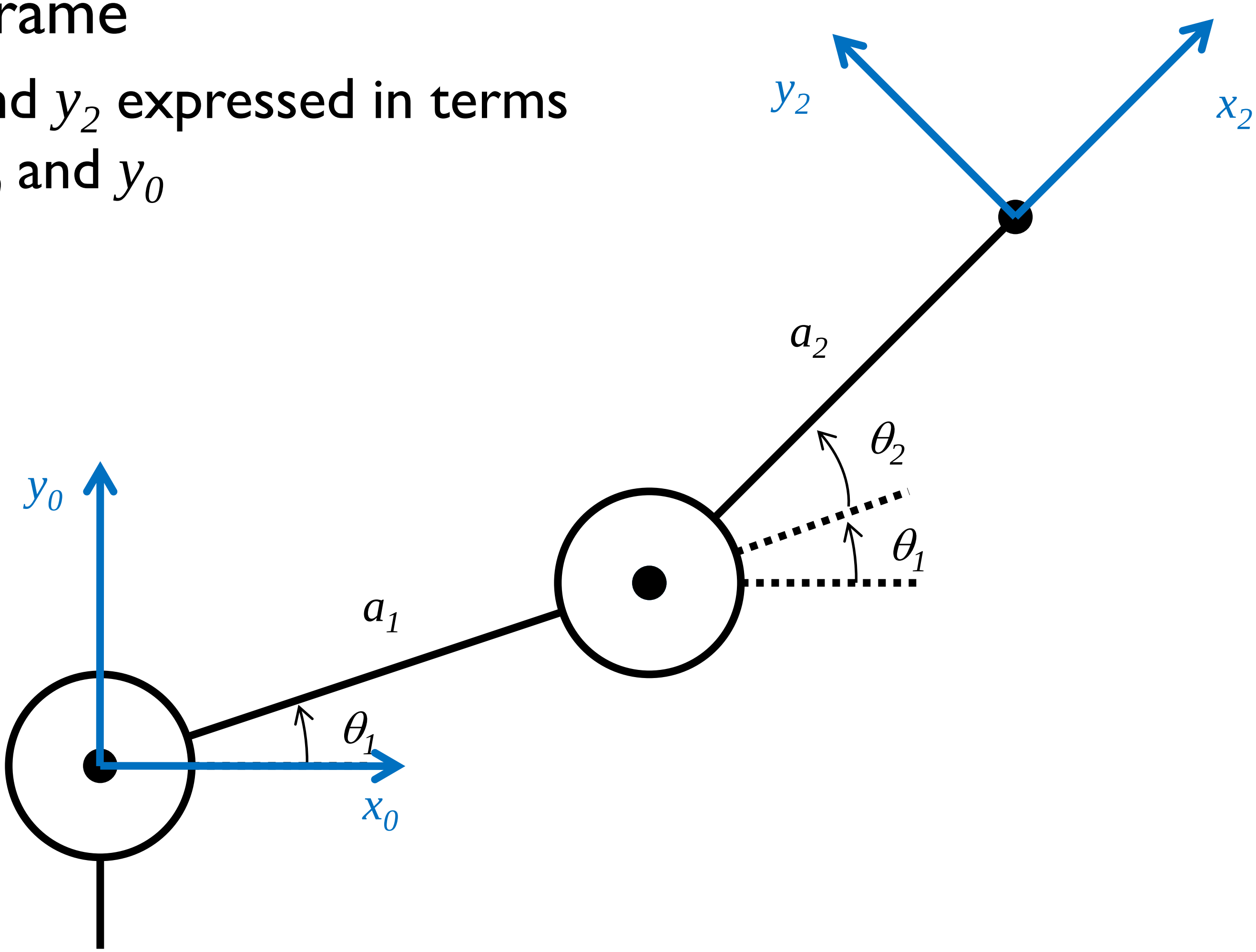
Forward Kinematics

- because the base frame and frame *1* have the same orientation, we can sum the coordinates to find the position of the end effector in the base frame



Forward Kinematics

- we also want the orientation of frame 2 with respect to the base frame
  - x_2 and y_2 expressed in terms of x_0 and y_0

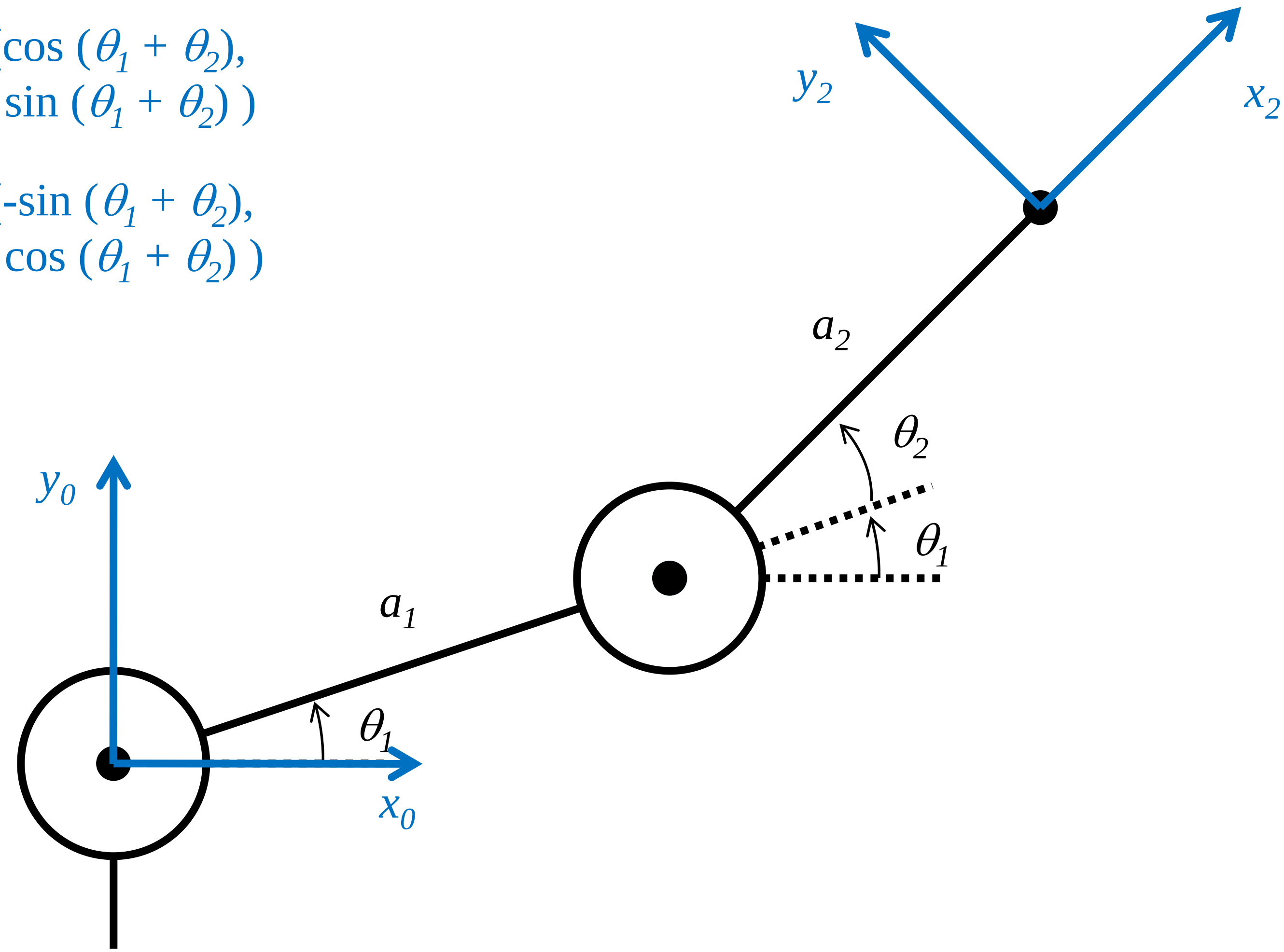


Forward Kinematics

- without proof I claim:

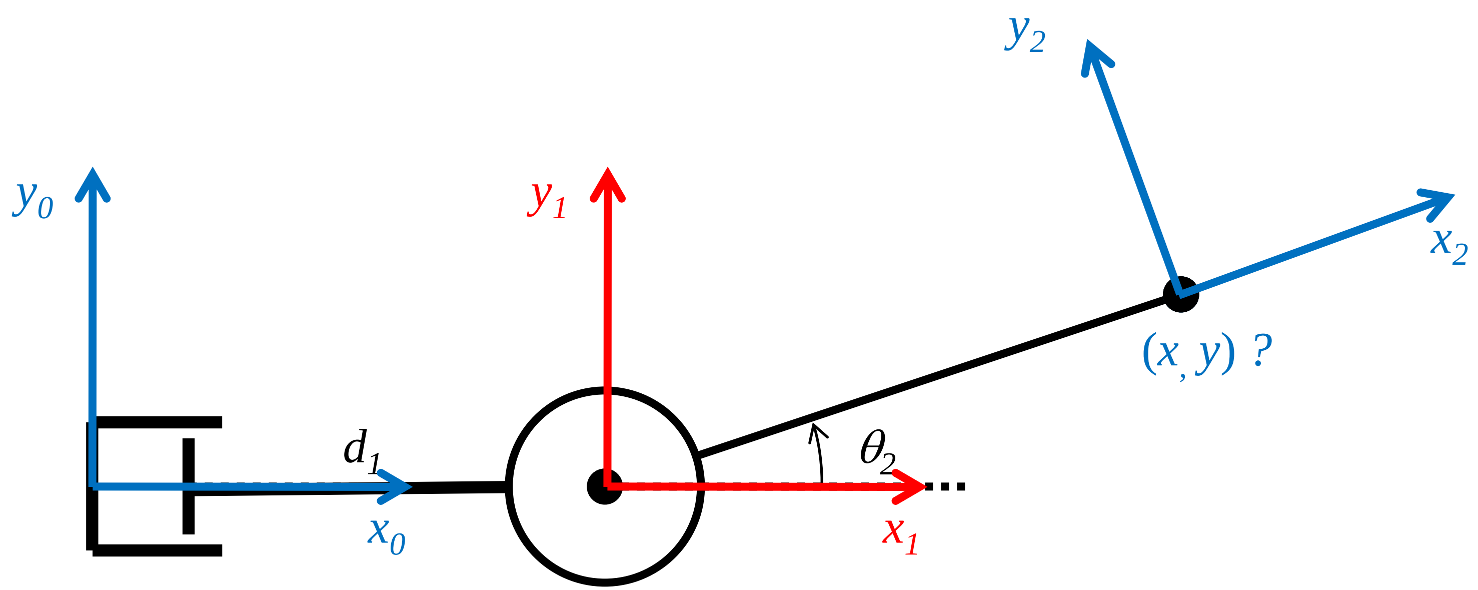
$$x_2 = (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2))$$

$$y_2 = (-\sin(\theta_1 + \theta_2), \cos(\theta_1 + \theta_2))$$



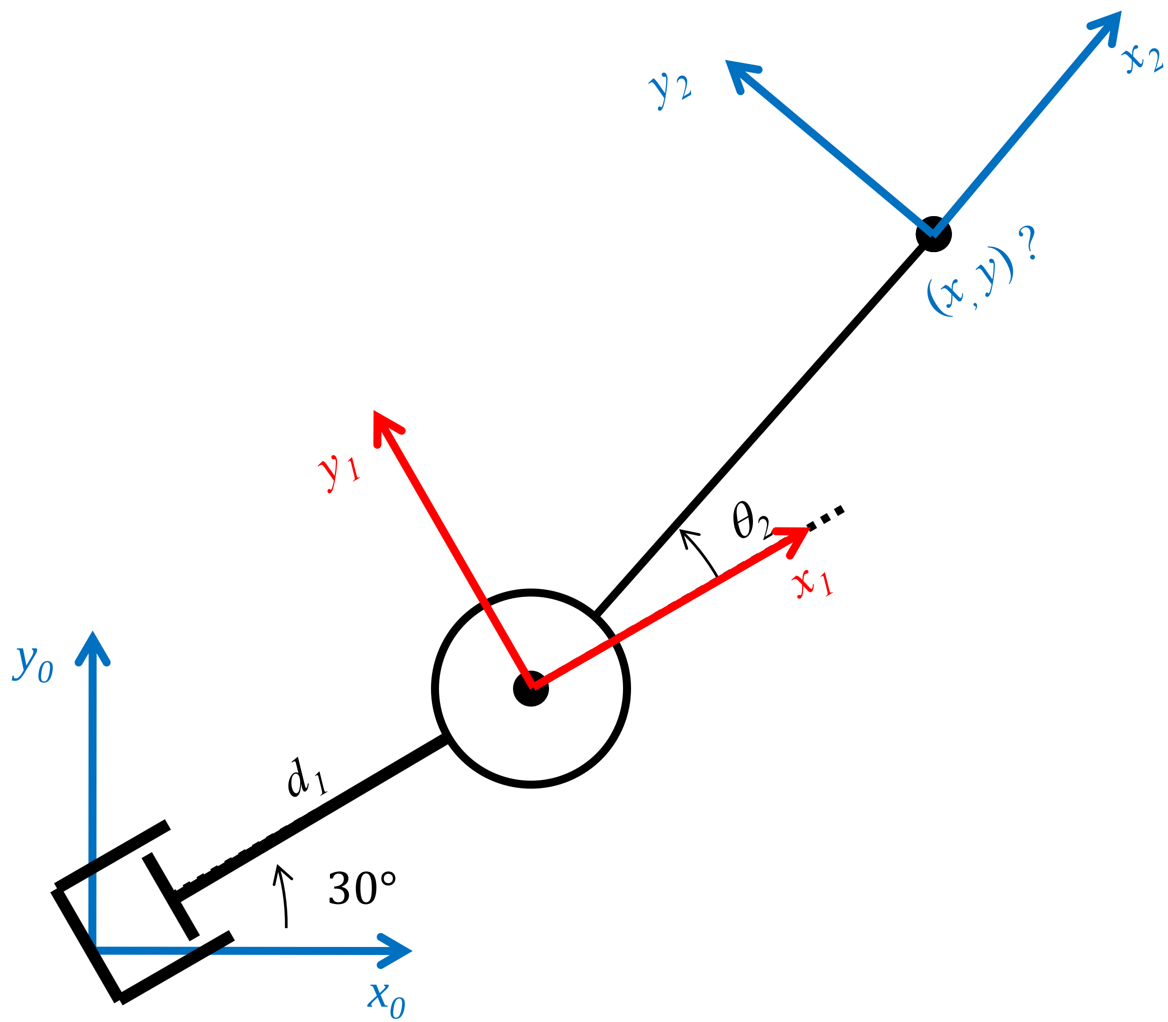
Forward Kinematics

- find (x, y), x_2, and y_2 expressed in frame 0



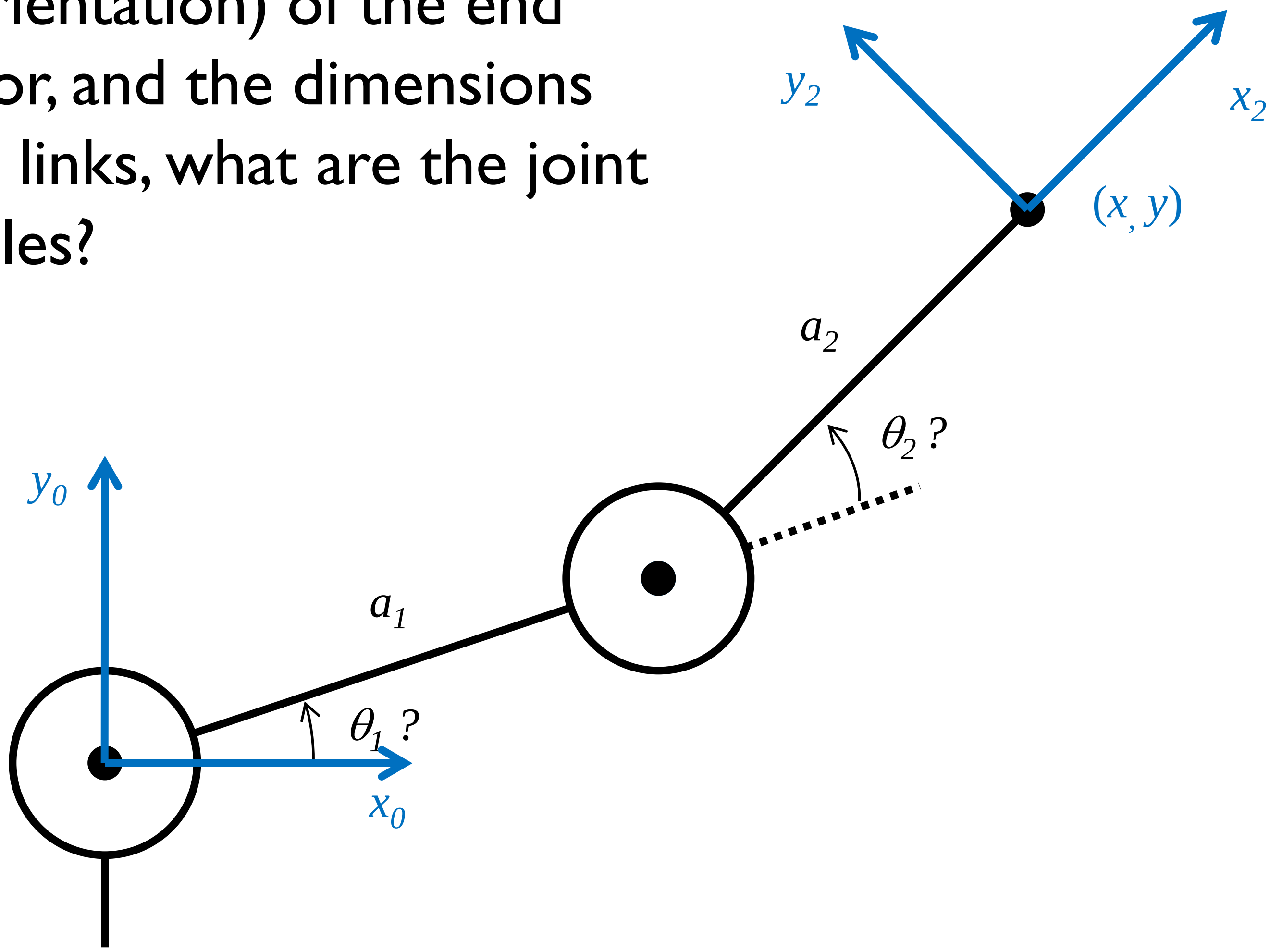
Forward Kinematics

- find (x, y), x_2, and y_2 expressed in frame 0



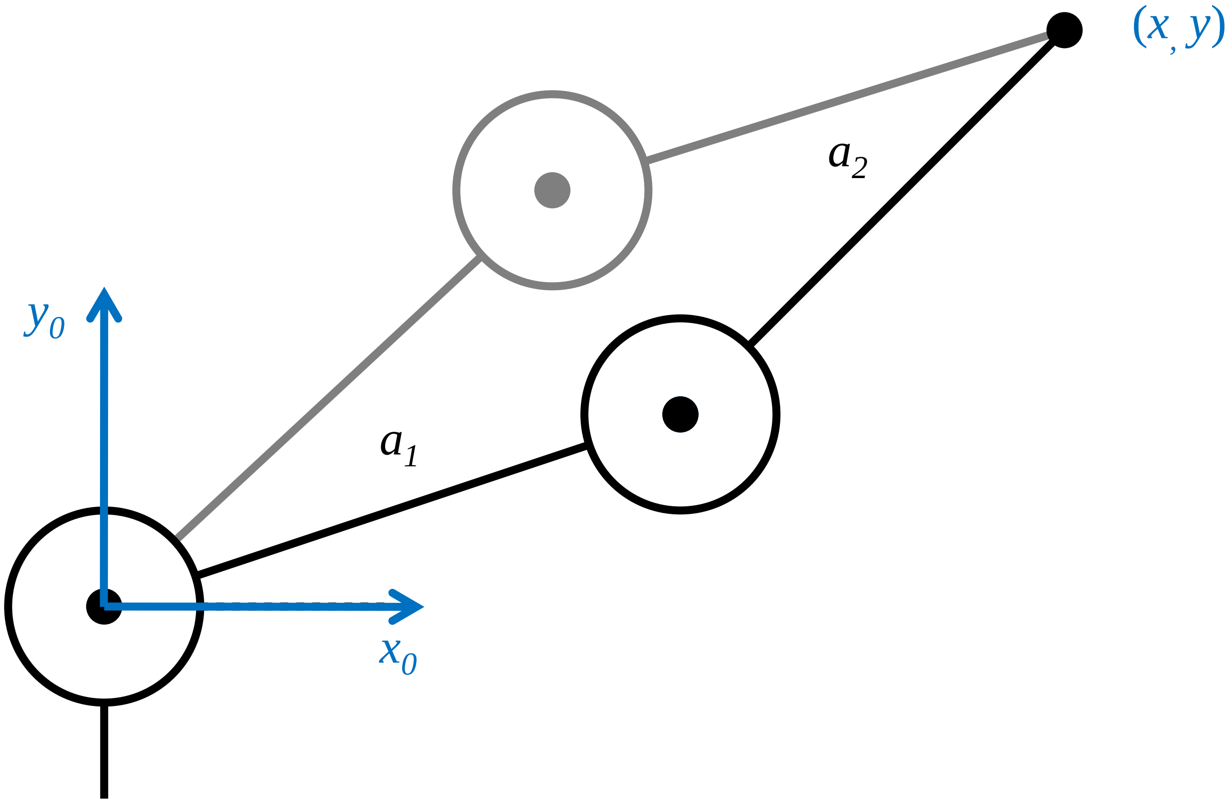
Inverse Kinematics

- given the position (and possibly the orientation) of the end effector, and the dimensions of the links, what are the joint variables?



Inverse Kinematics

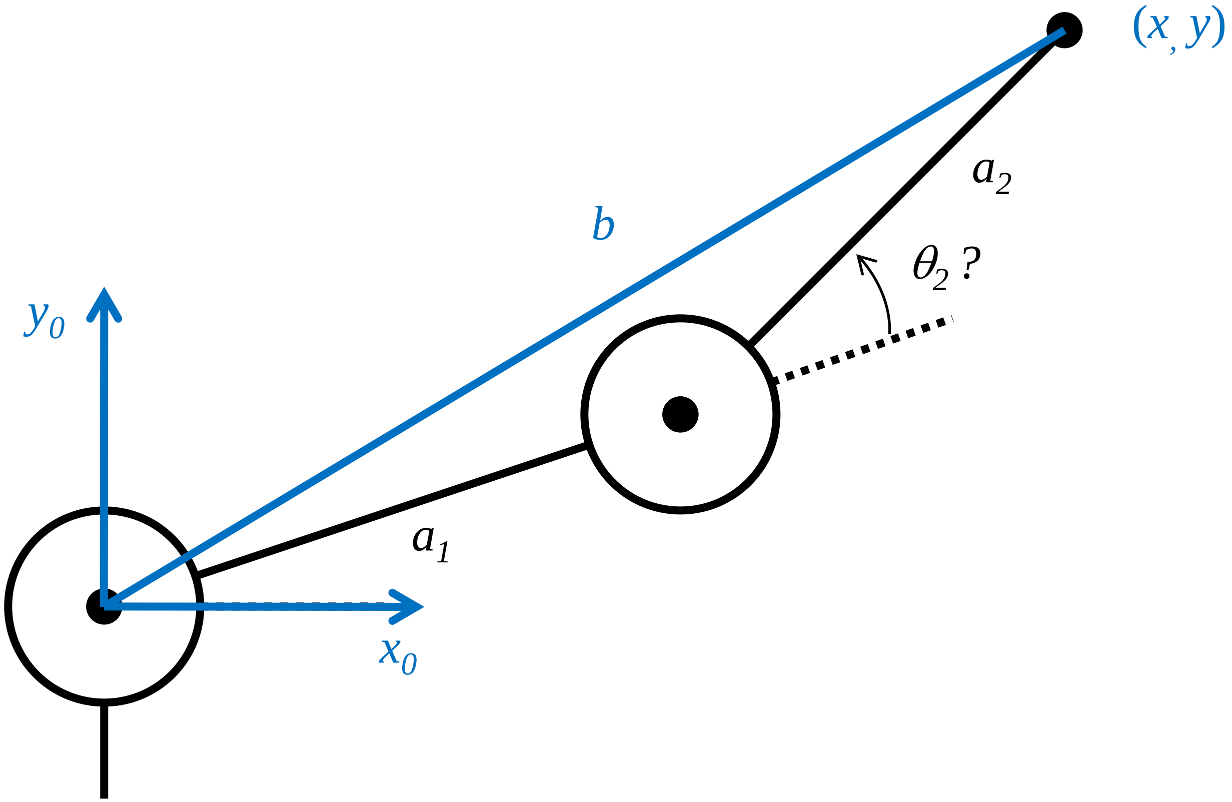
- harder than forward kinematics because there is often more than one possible solution



Inverse Kinematics

law of cosines

$$b^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2) = x^2 + y^2$$



Inverse Kinematics

$$-\cos(\pi-\theta_2)=\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

and we have the trigonometric identity

$$-\cos(\pi-\theta_2)=\cos(\theta_2)$$

therefore,

$$\cos\theta_2=\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}=C_2$$

We could take the inverse cosine, but this gives only one of the two solutions.

Inverse Kinematics

Instead, use the two trigonometric identities:

$$\sin^2\theta + \cos^2\theta_2 = 1 \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

to obtain

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-C_2^2}}{C_2}$$

which yields both solutions for θ_2. In many programming languages you would use the four quadrant inverse tangent function `atan2`

```
c2 = (x*x + y*y - a1*a1 - a2*a2) / (2*a1*a2);
s2 = sqrt(1 - c2*c2);
theta21 = atan2(s2, c2);
theta22 = atan2(-s2, c2);
```

Inverse Kinematics

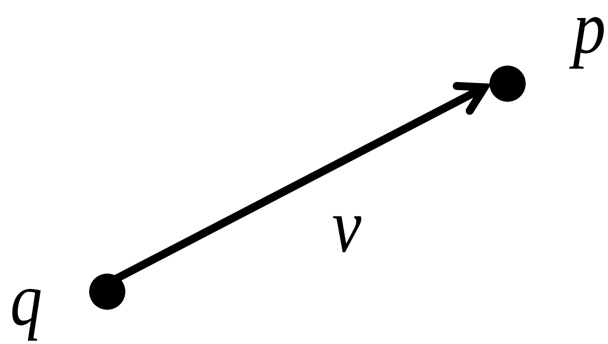
- Exercise for the student: show that

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}\right)$$

Spatial Descriptions

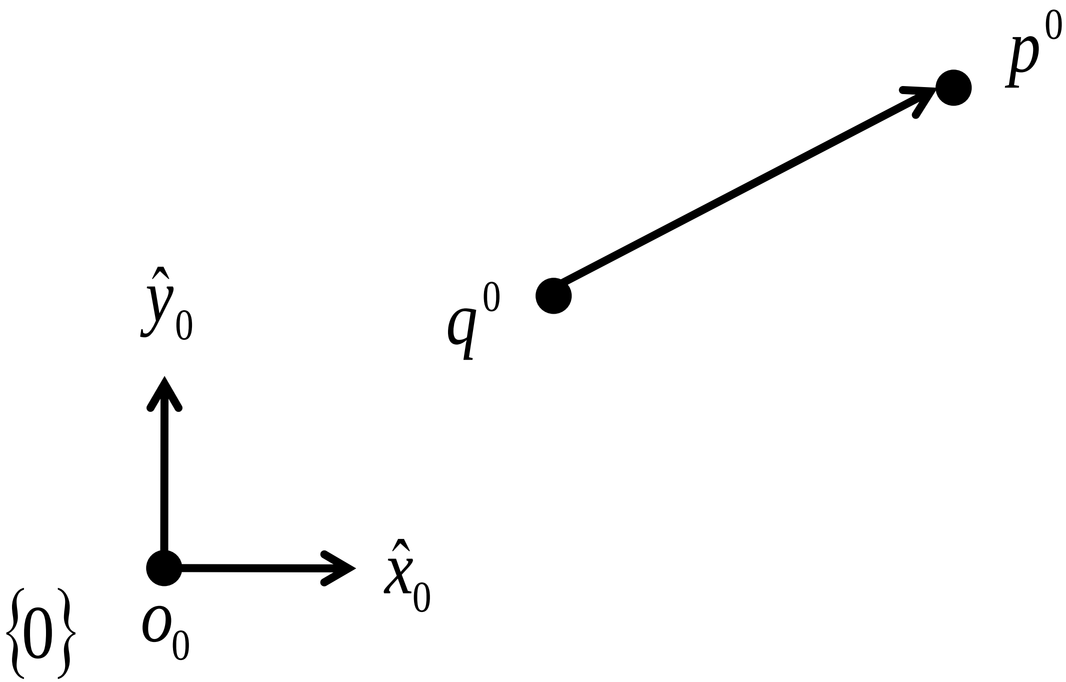
Points and Vectors

- point : a location in space
- vector : magnitude (length) and direction between two points



Coordinate Frames

- choosing a frame (a point and two perpendicular vectors of unit length) allows us to assign coordinates



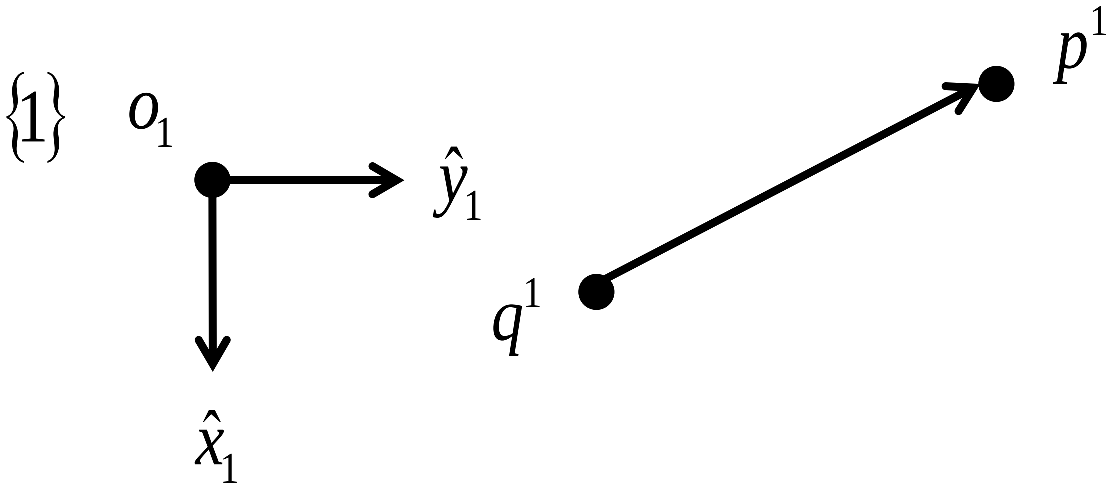
$$p^0 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

$$q^0 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

$$v^0 = p^0 - q^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Coordinate Frames

- the coordinates change depending on the choice of frame



$$p^1 = \begin{bmatrix} -0.5 \\ 4 \end{bmatrix}$$

$$q^1 = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$$

$$v^1 = p^1 - q^1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

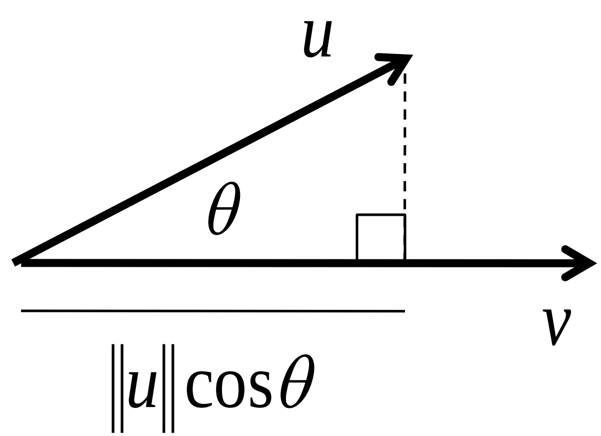
Dot Product

- the dot product of two vectors

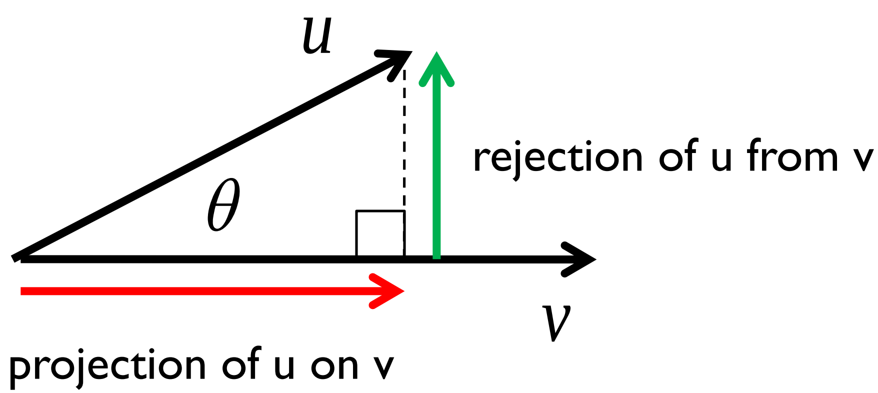
$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = u^T v$$

$$u \cdot v = \|u\| \|v\| \cos\theta$$



Vector Projection and Rejection



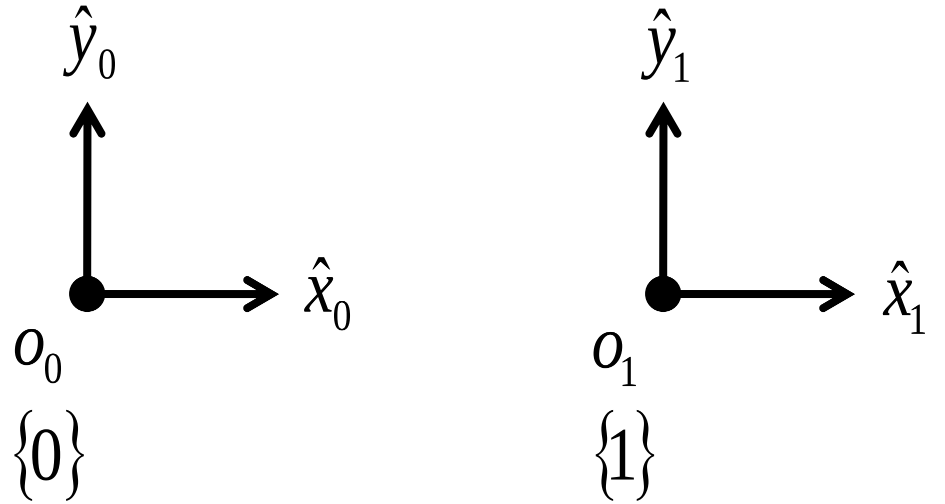
$$u - \frac{u \cdot v}{v \cdot v} v$$

$$\frac{u \cdot v}{v \cdot v} v$$

- if u and v are unit vectors (have magnitude equal to 1) then the projection becomes

$$\hat{u} \cdot \hat{v}\, \hat{v}$$

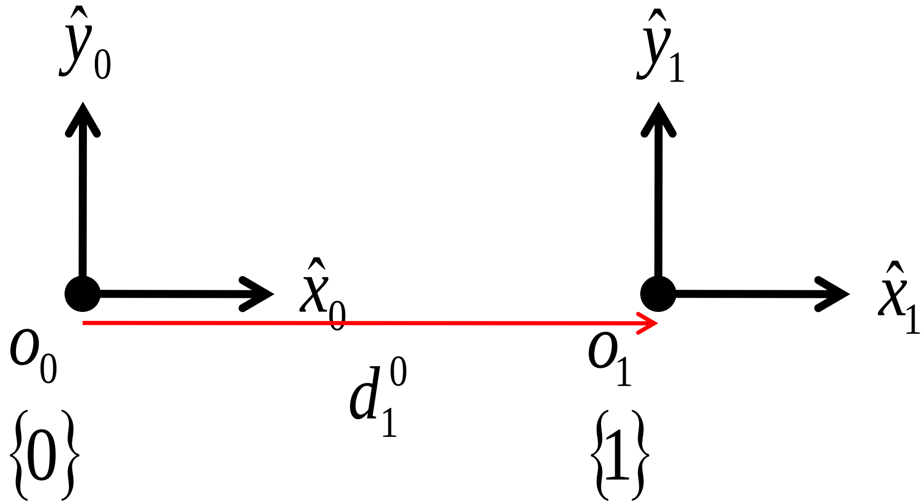
Translation



- suppose we are given o_1 expressed in $\{0\}$

$$o_1^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Translation 1



- the location of $\{1\}$ expressed in $\{0\}$

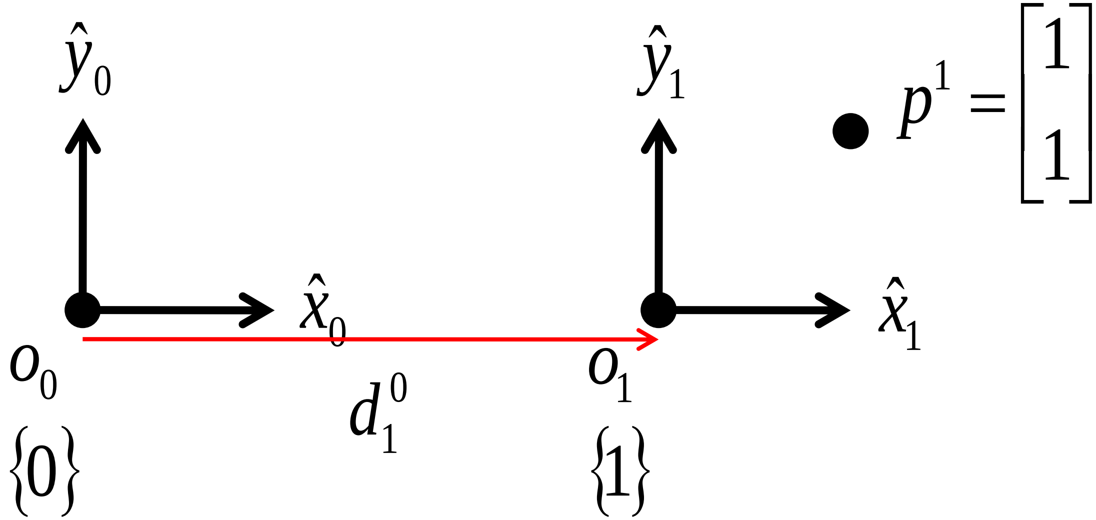
$$d_1^0 = o_1^0 - o_0^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Translation 1

1. the translation vector d_j^i can be interpreted as the location of frame {j} expressed in frame {i}

Translation 2

a point expressed in frame $\{1\}$



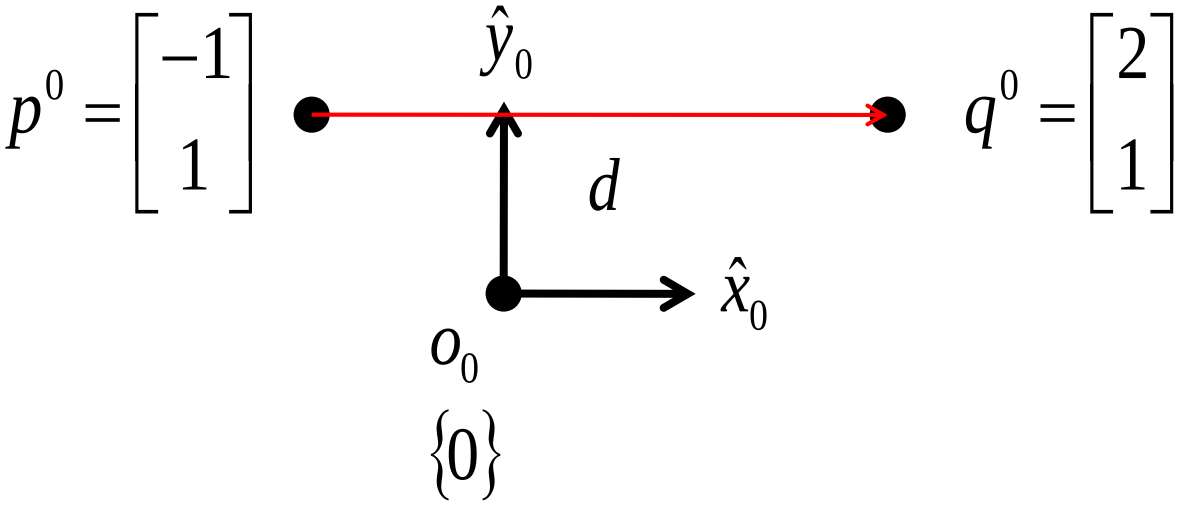
- p^1 expressed in $\{0\}$

$$p^0 = d_1^0 + p^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Translation 2

2. the translation vector d_j^i can be interpreted as a coordinate transformation of a point from frame {j} to frame {i}

Translation 3



- q^0 expressed in $\{0\}$

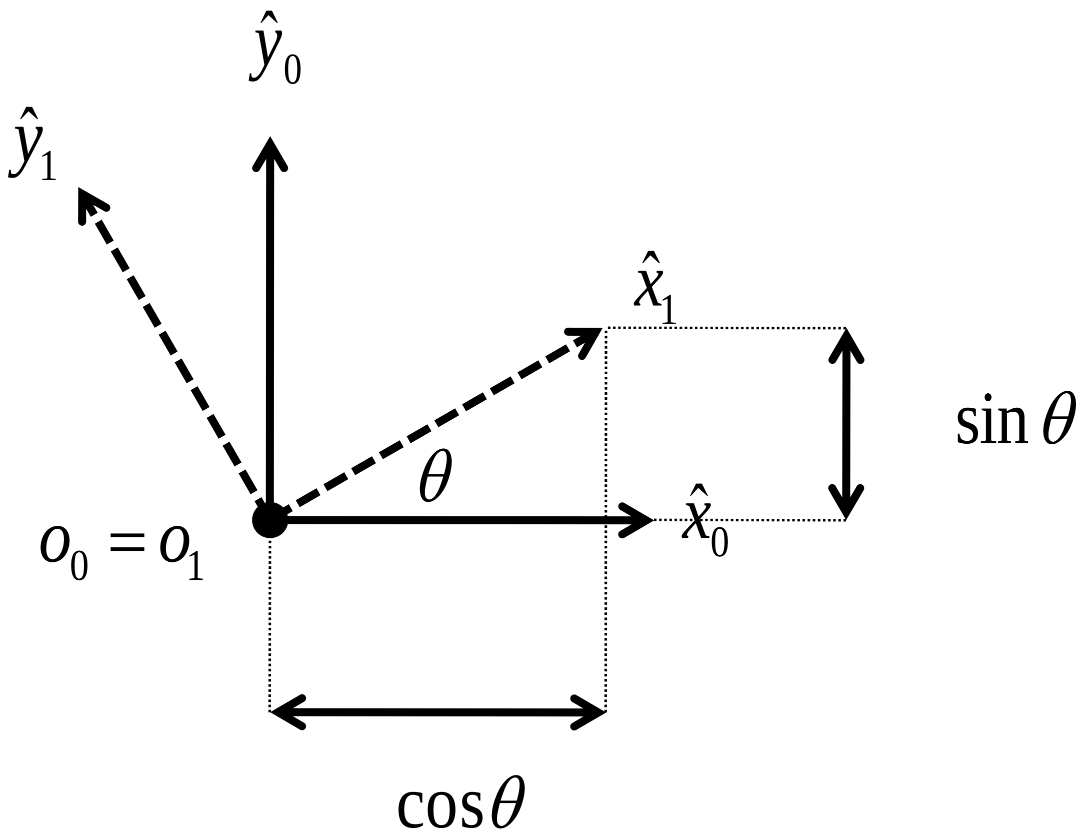
$$q^0 = d + p^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Translation 3

3. the translation vector d can be interpreted as an operator that takes a point and moves it to a new point in the same frame

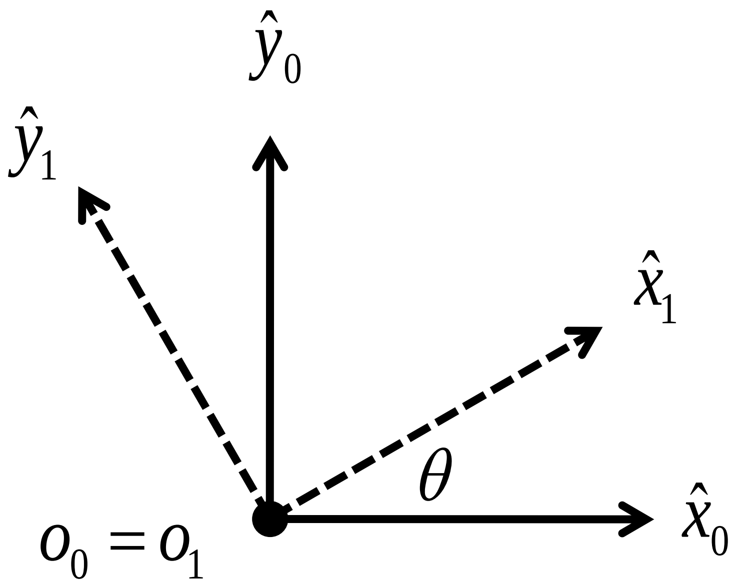
Rotation

- suppose that frame *{1}* is rotated relative to frame *{0}*



Rotation 1

- the orientation of frame {1} expressed in {0}



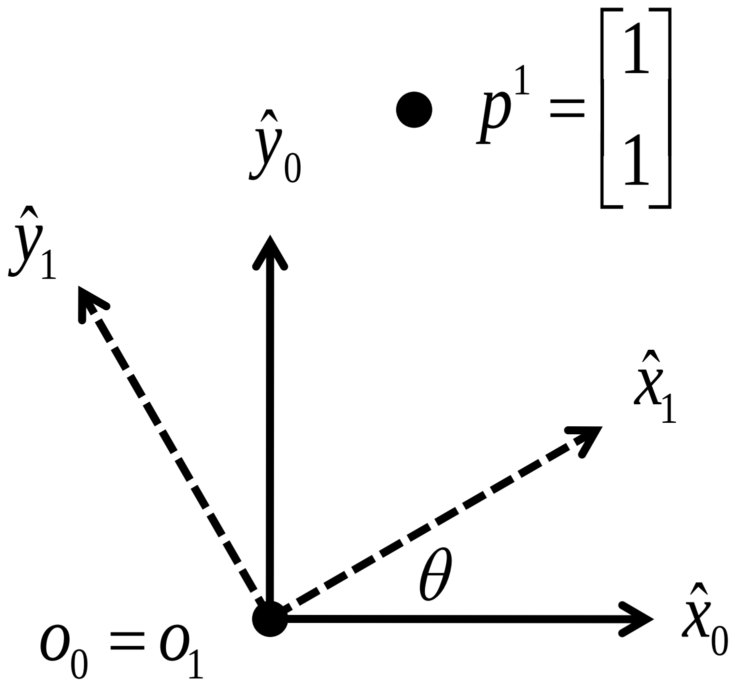
$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

Rotation 1

1. the rotation matrix R_j^i can be interpreted as the orientation of frame {j} expressed in frame {i}

Rotation 2

- p^1 expressed in $\{0\}$



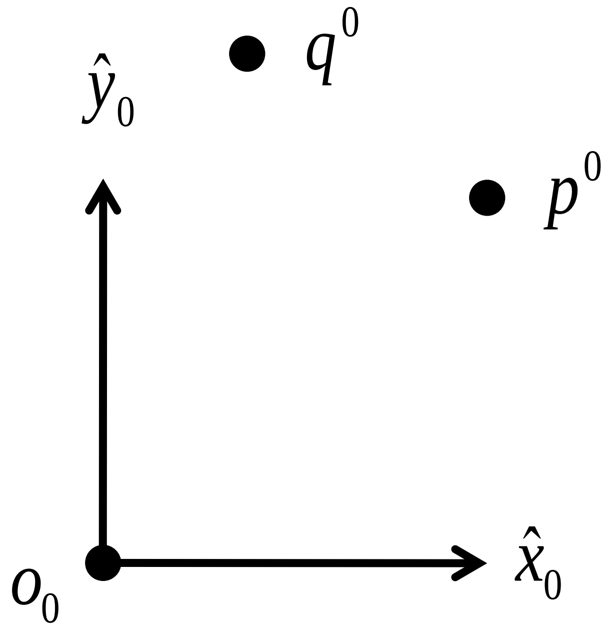
$$p^0 = R_1^0 p^1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Rotation 2

2. the rotation matrix R_j^i can be interpreted as a coordinate transformation of a point from frame {j} to frame {i}

Rotation 3

- q^0 expressed in $\{0\}$



$$q^0 = R\ p^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

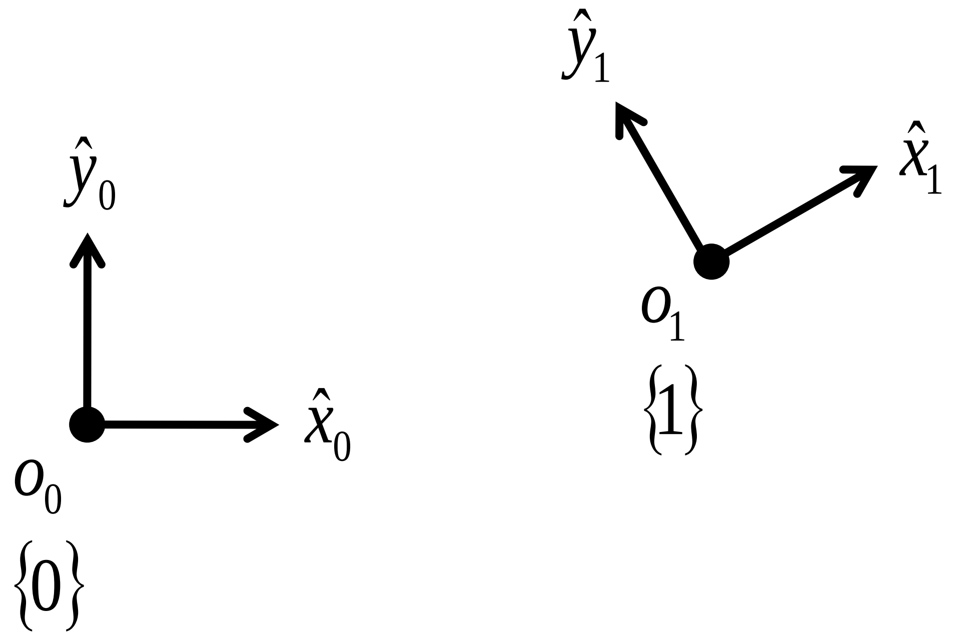
Rotation 3

3. the rotation matrix R can be interpreted as an operator that takes a point and moves it to a new point in the same frame

Properties of Rotation Matrices

- $R^T = R^{-1}$
- the columns of R are mutually orthogonal
- each column of R is a unit vector
- $\det R = 1$ (the determinant is equal to 1)

Rotation and Translation



Rotations in 3D

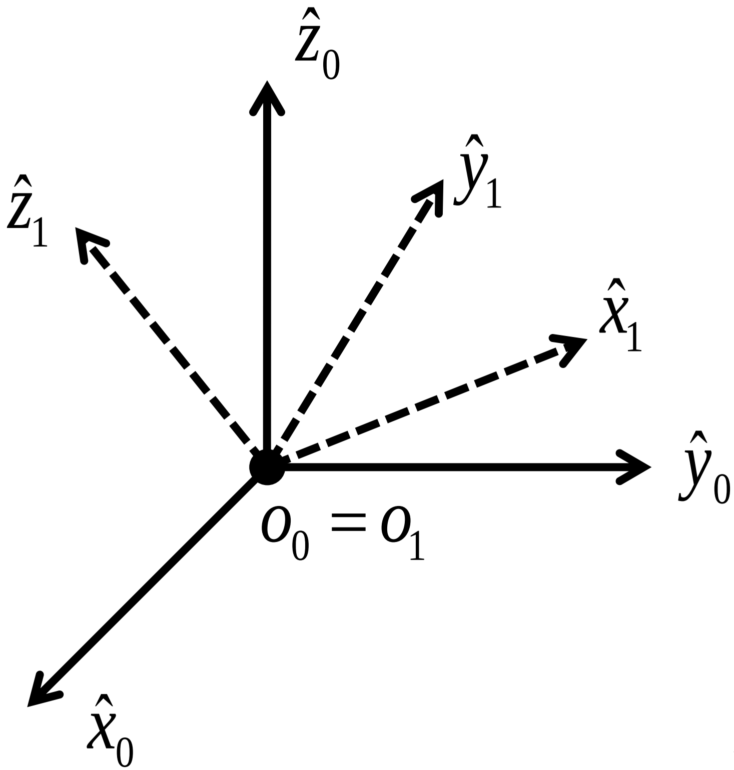
$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Rotations

Properties of Rotation Matrices

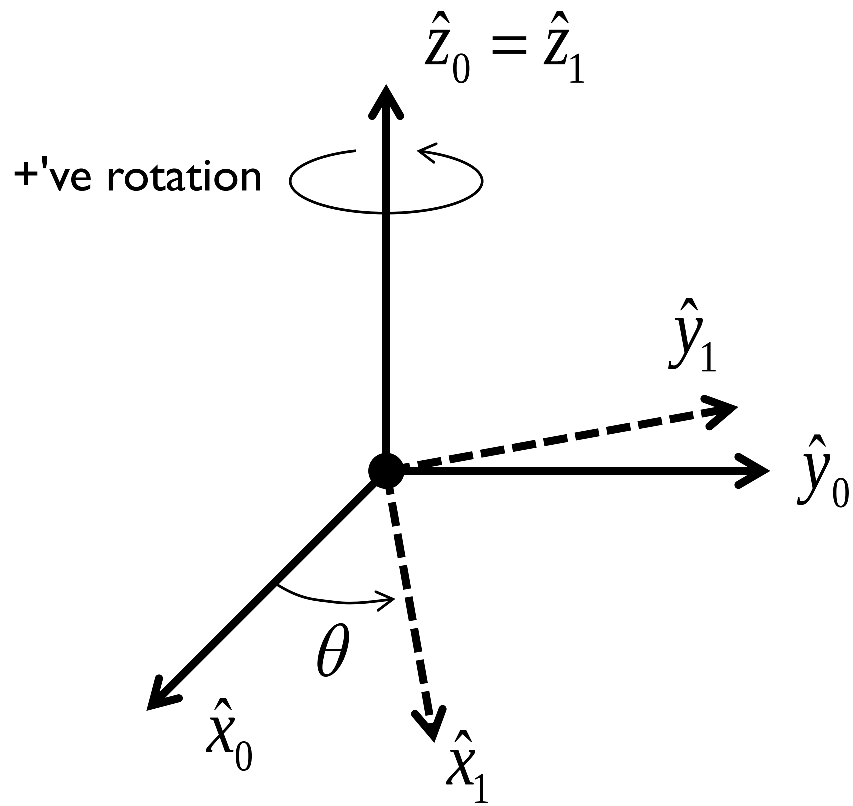
- $R^T = R^{-1}$
- the columns of R are mutually orthogonal
- each column of R is a unit vector
- $\det R = 1$ (the determinant is equal to 1)

Rotations in 3D

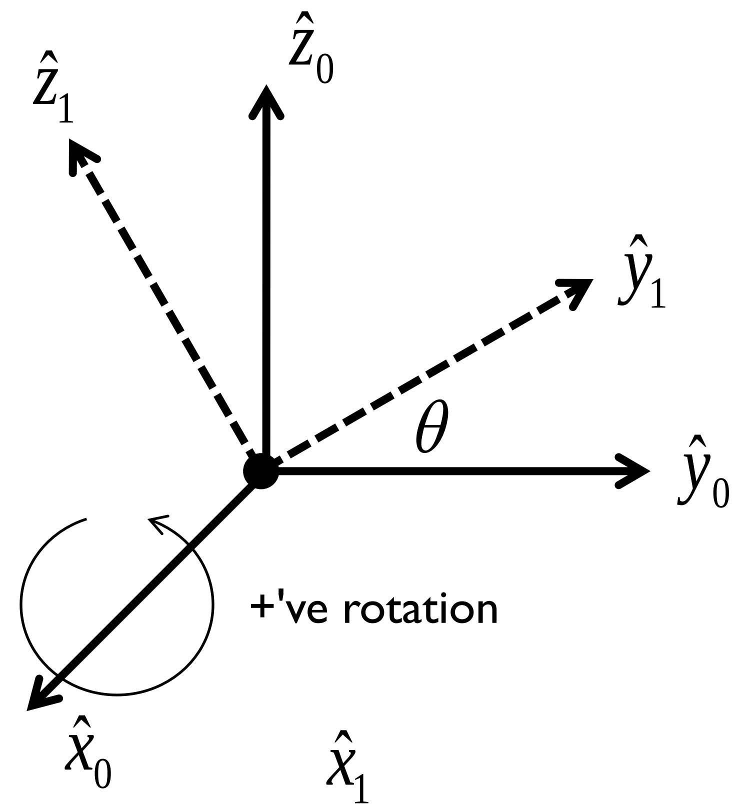


$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

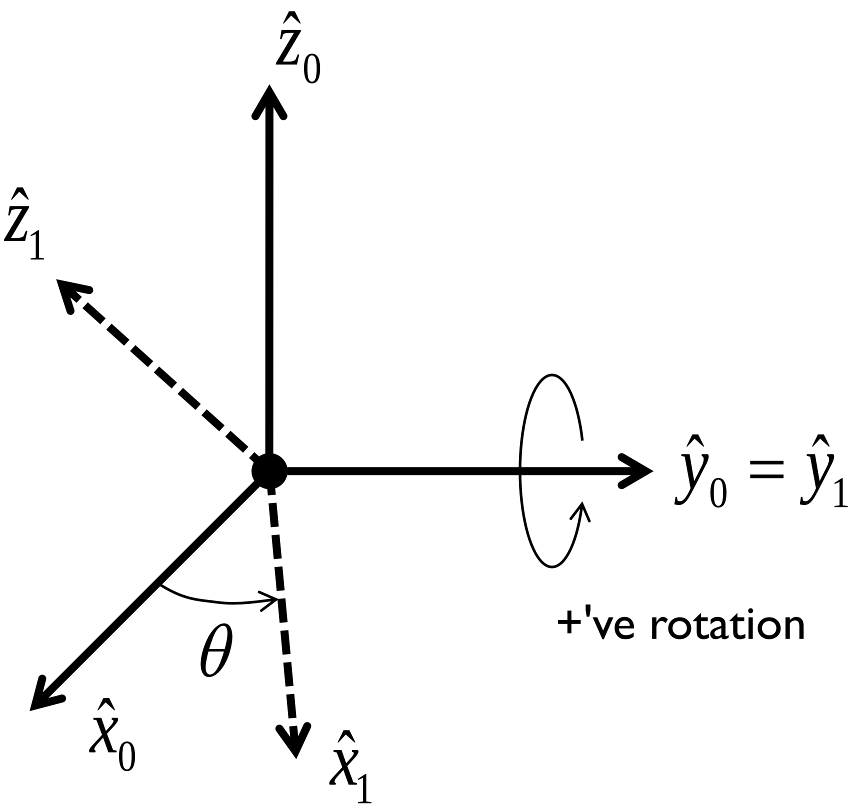
Rotation About *z*-axis



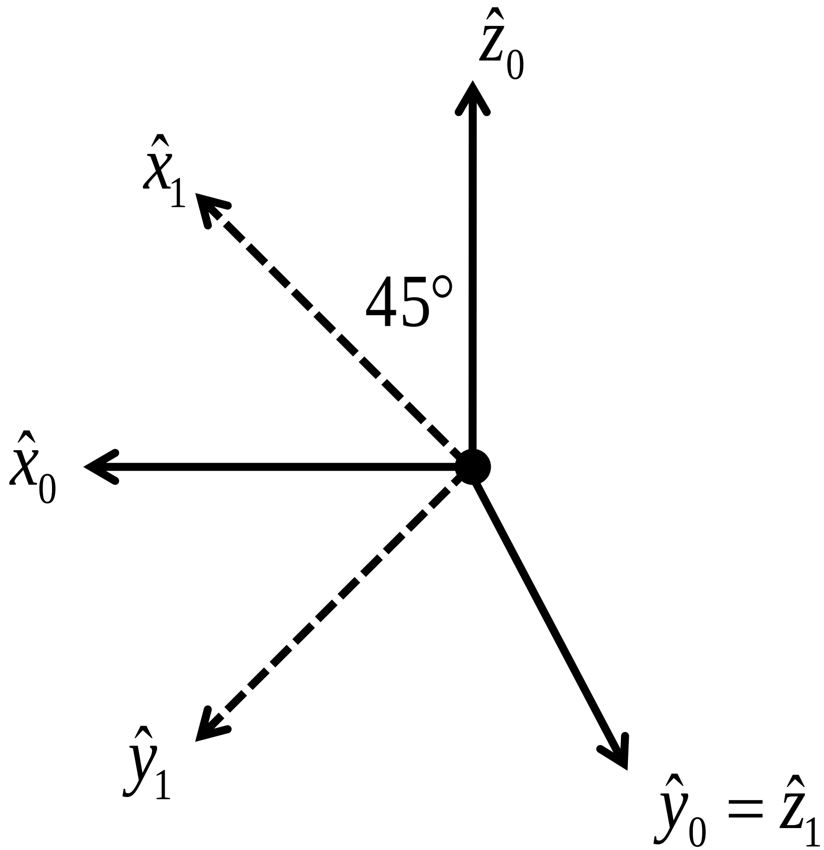
Rotation About x-axis



Rotation About y-axis



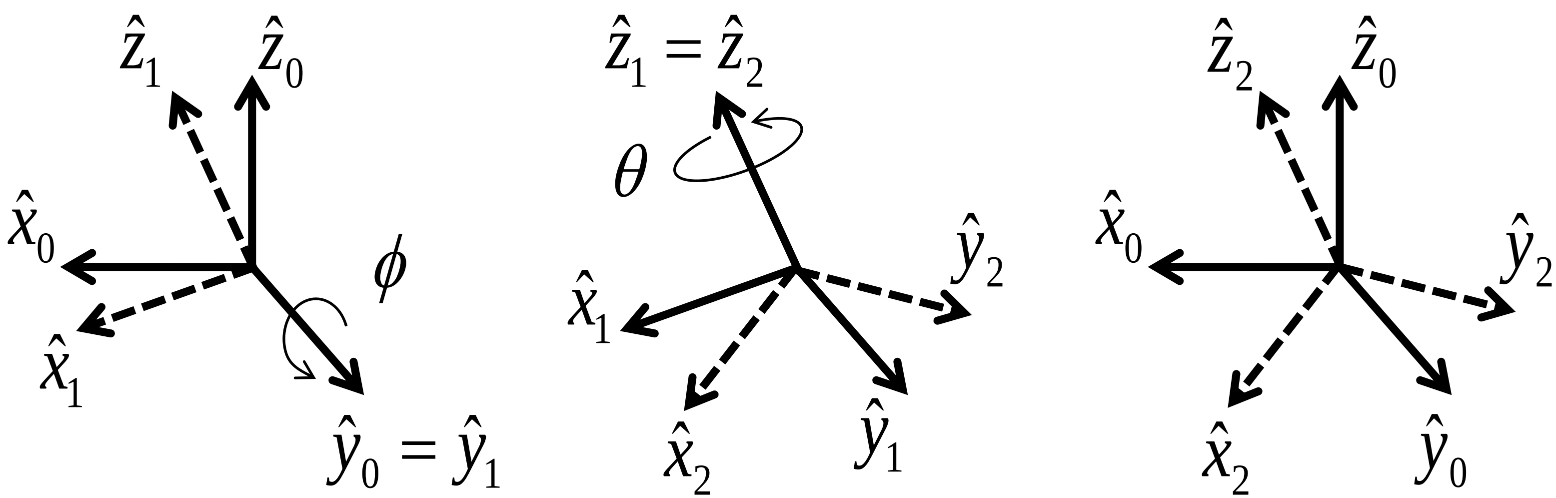
Relative Orientation Example



Successive Rotations in Moving Frames

1. Suppose you perform a rotation in frame {0} to obtain {1}.
2. Then you perform a rotation in frame {1} to obtain {2}.

What is the orientation of {2} relative to {0}?

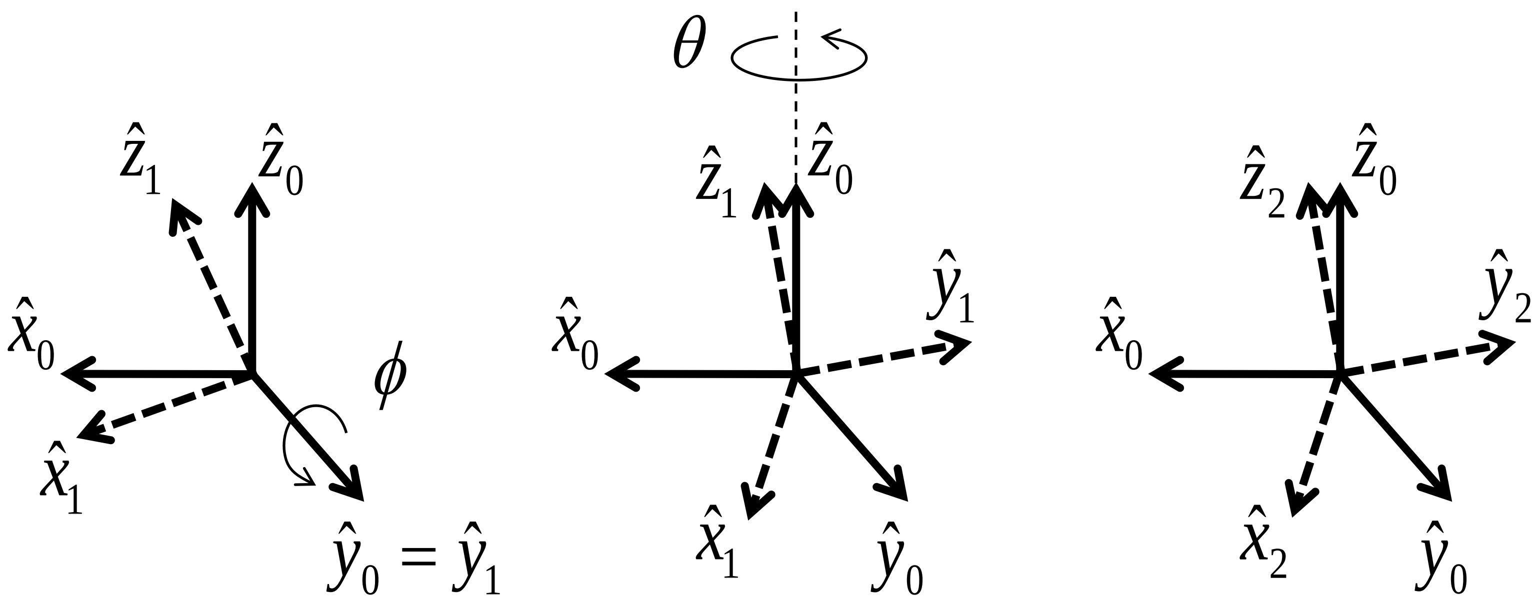


$R_1^0 = R_{y,\phi}$ $\quad$ $R_2^1 = R_{z,\theta}$ $\quad$ $R_2^0 = ?$

Successive Rotations in a Fixed Frame

1. Suppose you perform a rotation in frame {0} to obtain {1}.
2. Then you rotate {1} in frame {0} to obtain {2}.

What is the orientation of {2} relative to {0}?



$R_1^0 = R_{y,\phi}$ $\quad$ $R = R_{z,\theta}$ $\quad$ $R_2^0 = ?$

Composition of Rotations

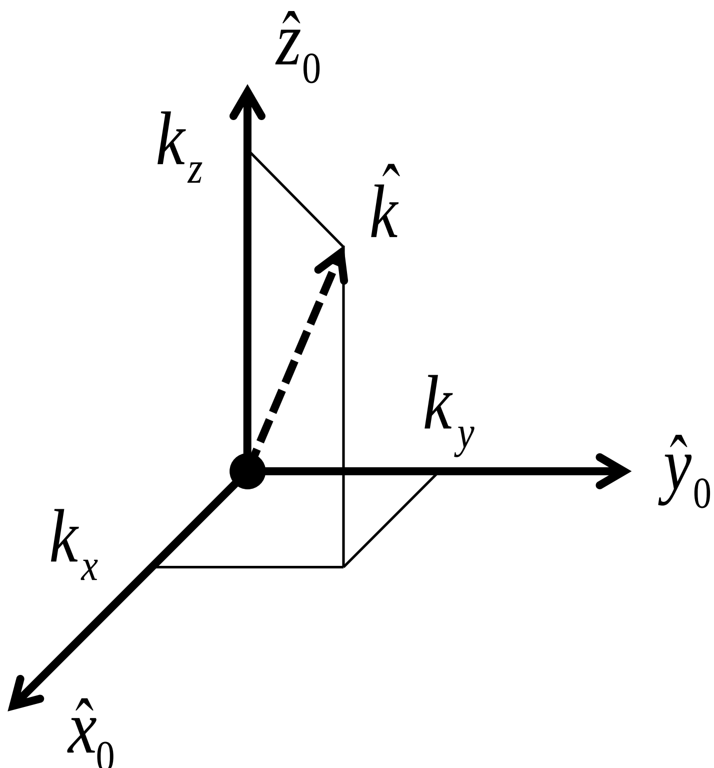
1. Given a fixed frame {0} and a current frame {1} and R_1^0
   - if {2} is obtained by a rotation R in the *current frame* {1} then use postmulitplication to obtain:

$$R = R_2^1 \quad \text{and} \quad R_2^0 = R_1^0 R_2^1$$

2. Given a fixed frame {0} and a frame {1} and
   - if {2} is obtained by a rotation R in the *fixed frame* {0} then use premultiplication to obtain:

$$R_2^0 = R R_1^0$$

Rotation About a Unit Axis



$$c_\theta = \cos\theta$$
$$s_\theta = \sin\theta$$
$$v_\theta = 1 - \cos\theta$$

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Rigid Body Transformations

Homogeneous Representation

- translation represented by a vector d
  - vector addition
- rotation represented by a matrix R
  - matrix-matrix and matrix-vector multiplication
- convenient to have a uniform representation of translation and rotation
  - obviously vector addition will not work for rotation
  - can we use matrix multiplication to represent translation?

Homogeneous Representation

- consider moving a point p by a translation vector d

$$p + d = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} p_x + d_x \\ p_y + d_y \\ p_z + d_z \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} p_x + d_x \\ p_y + d_y \\ p_z + d_z \end{bmatrix}$$

not possible as matrix-vector multiplication always leaves the origin unchanged

Homogeneous Representation

- consider an augmented vector p_h and an augmented matrix D

$$p_h = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Dp_h = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + d_x \\ p_y + d_y \\ p_z + d_z \\ 1 \end{bmatrix}$$

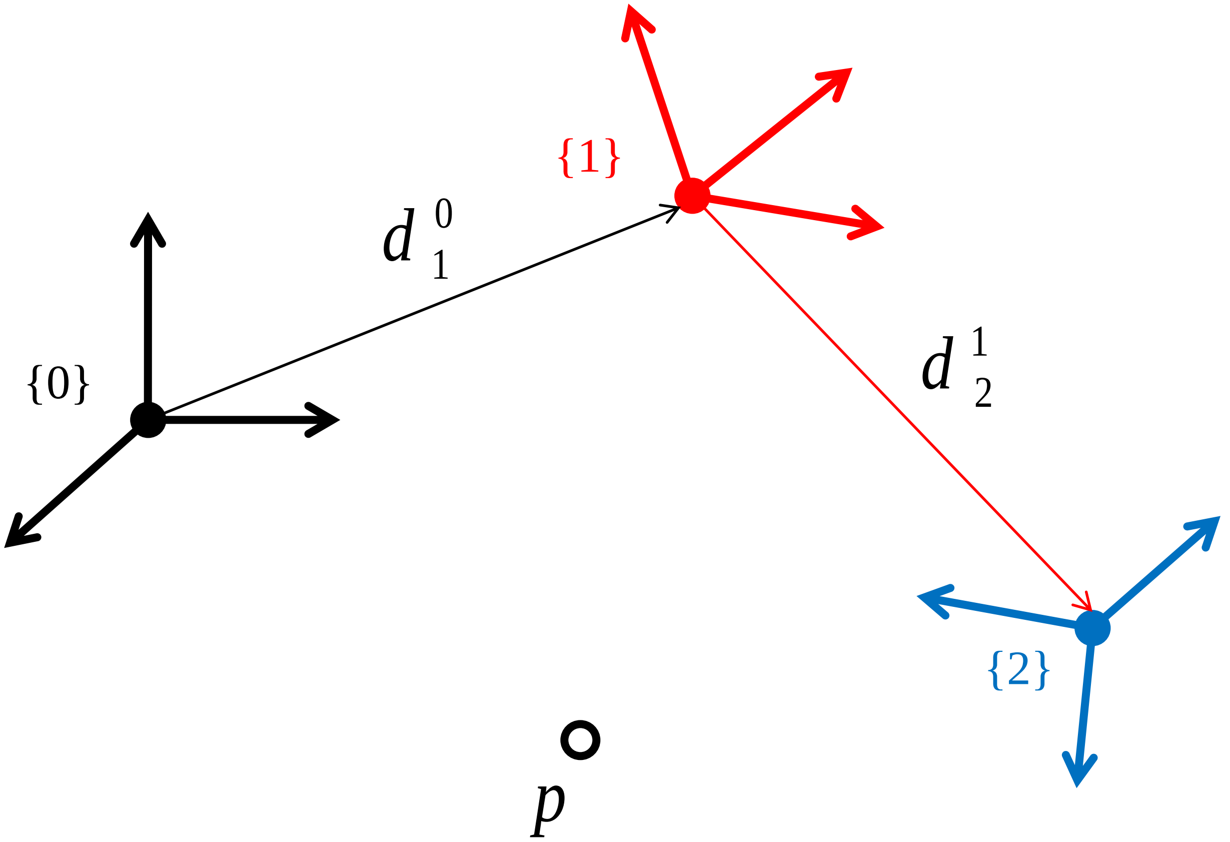
Homogeneous Representation

- the augmented form of a rotation matrix R_{3x3}

$$R = \begin{bmatrix} \begin{bmatrix} & & \\ & R_{3x3} & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

$$Rp_h = \begin{bmatrix} \begin{bmatrix} & & \\ & R_{3x3} & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} R_{3x3}\, p \end{bmatrix} \\ 1 \end{bmatrix}$$

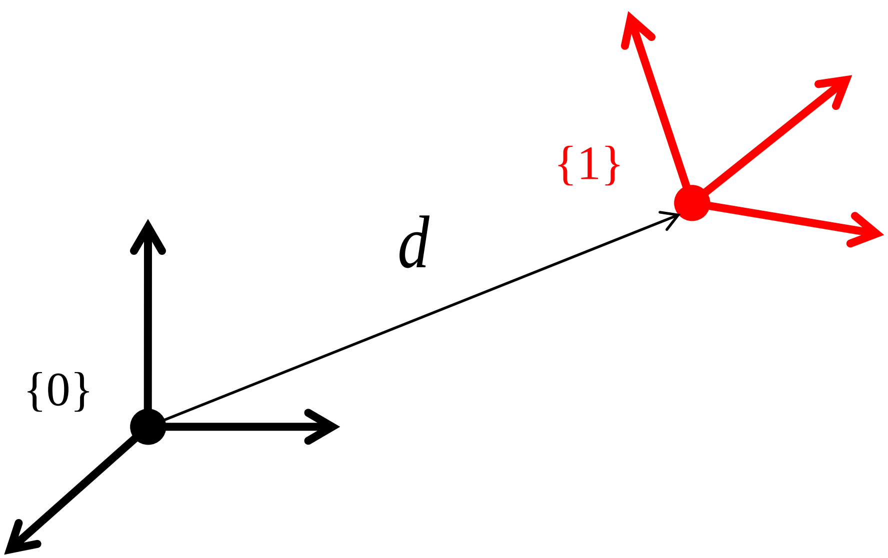
Rigid Body Transformations in 3D



Rigid Body Transformations in 3D

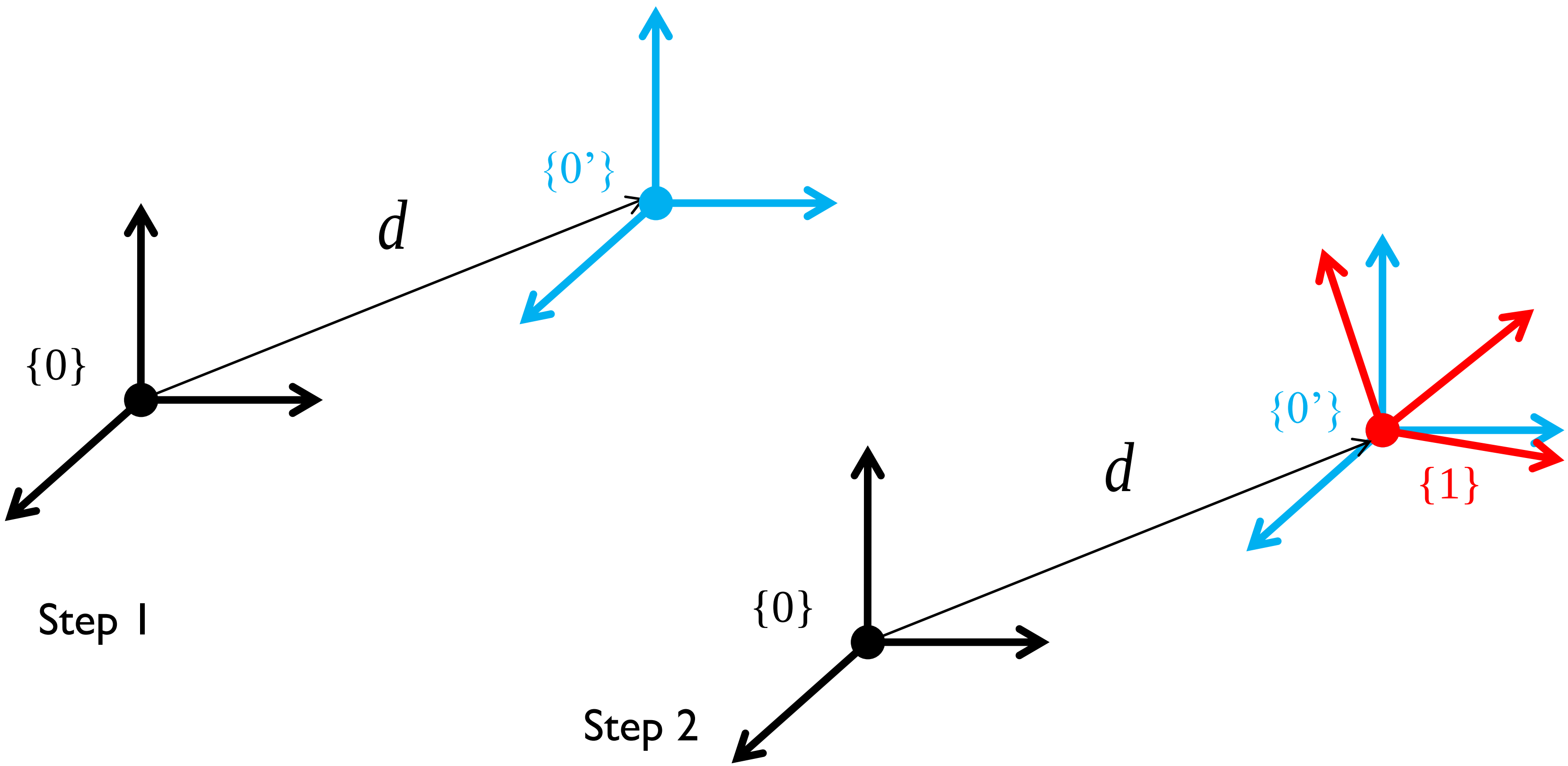
- suppose {1} is a rotated and translated relative to {0}
- what is the pose (the orientation and position) of {1} expressed in {0} ?

$$T_1^0 = ?$$



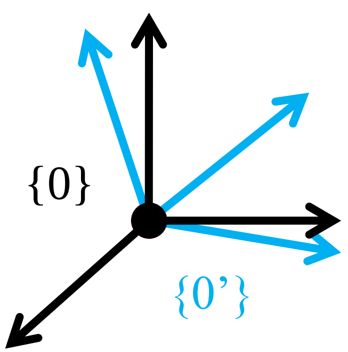
Rigid Body Transformations in 3D

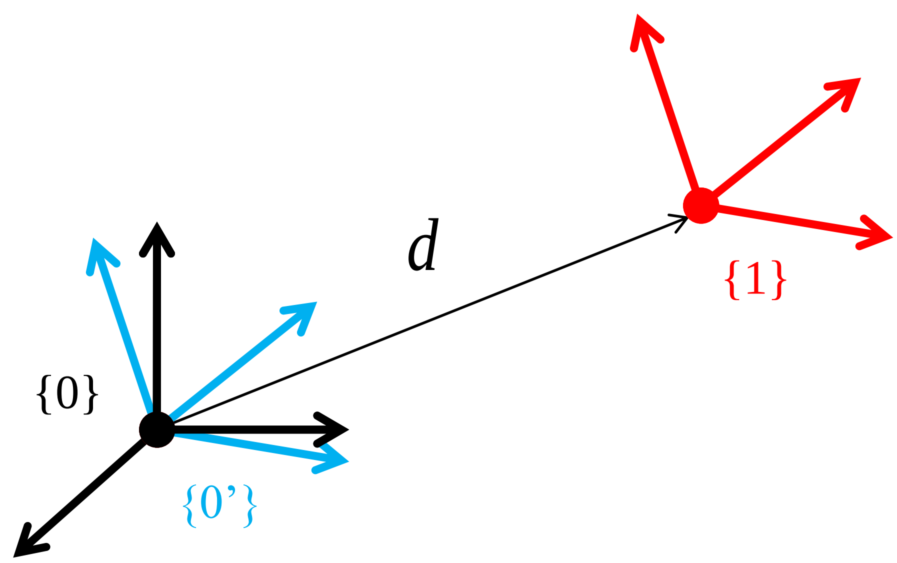
- suppose we use the moving frame interpretation (postmultiply transformation matrices)
  1. translate in {0} to get {0'} $D_{0'}^{0}$
  2. and then rotate in {0'} to get {1} $D_{0'}^{0} R_{1}^{0'}$



Rigid Body Transformations in 3D

- suppose we use the fixed frame interpretation (premultiply transformation matrices)
  1. rotate in $\{0\}$ to get $\{0'\}$ $\quad R$
  2. and then translate in $\{0\}$ in to get $\{1\}$ $\quad DR$





Rigid Body Transformations in 3D

- both interpretations yield the same transformation

$$T_1^0 = D\,R$$

$$= \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & R_{3x3} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & & & \\ & R_{3x3} & & d \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Representation

- every rigid-body transformation can be represented as a rotation followed by a translation *in the same frame*
  - as a 4x4 matrix

$$T = \begin{bmatrix} & R & & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where R is a 3x3 rotation matrix and d is a 3x1 translation vector

Homogeneous Representation

- in some frame i
  - points

$$P^i = \begin{bmatrix} p^i \\ 1 \end{bmatrix}$$

  - vectors

$$V^i = \begin{bmatrix} v^i \\ 0 \end{bmatrix}$$

Inverse Transformation

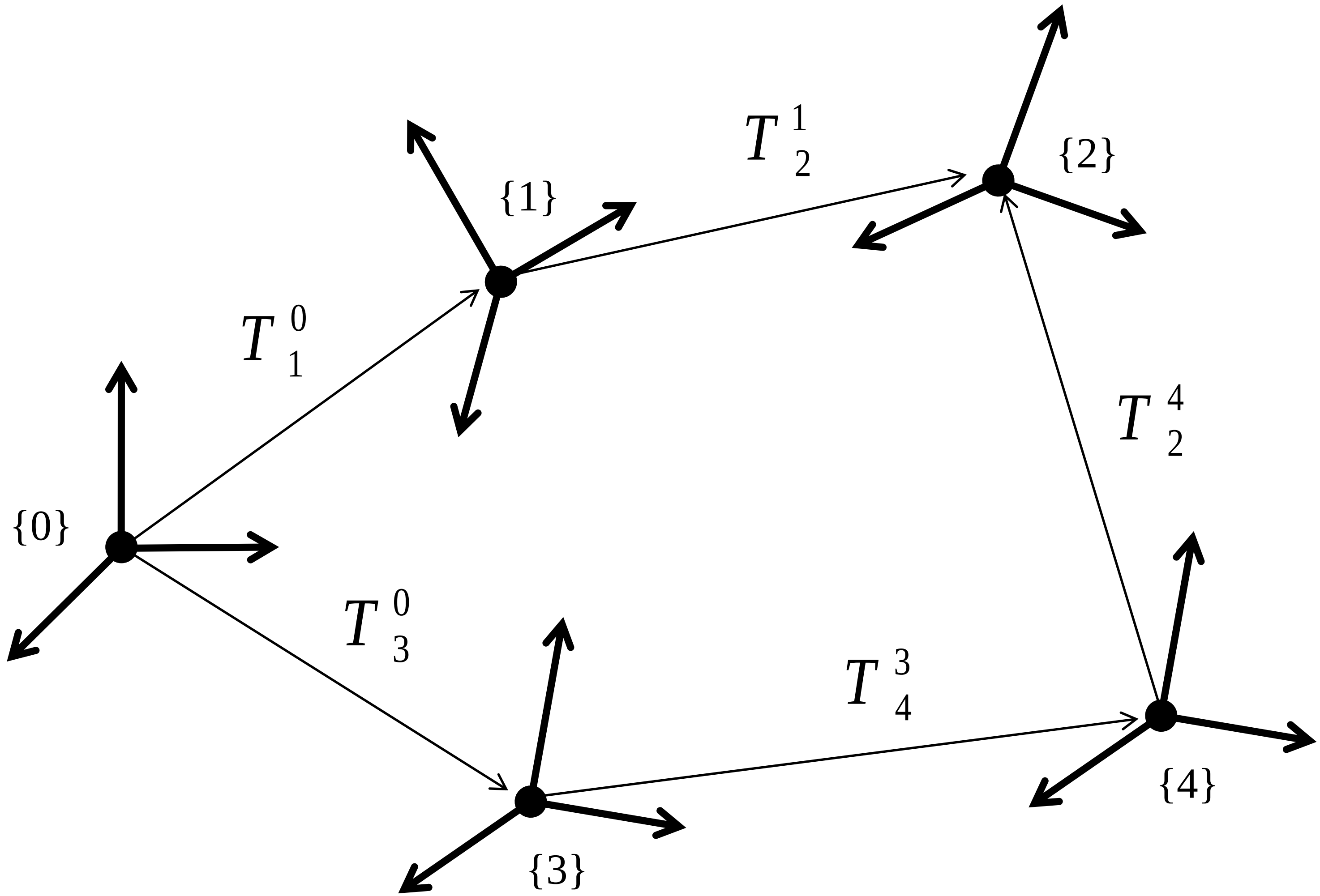
- the inverse of a transformation undoes the original transformation
  - if

$$T = \begin{bmatrix} & R & & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - then

$$T^{-1} = \begin{bmatrix} & R^T & & -R^T d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform Equations



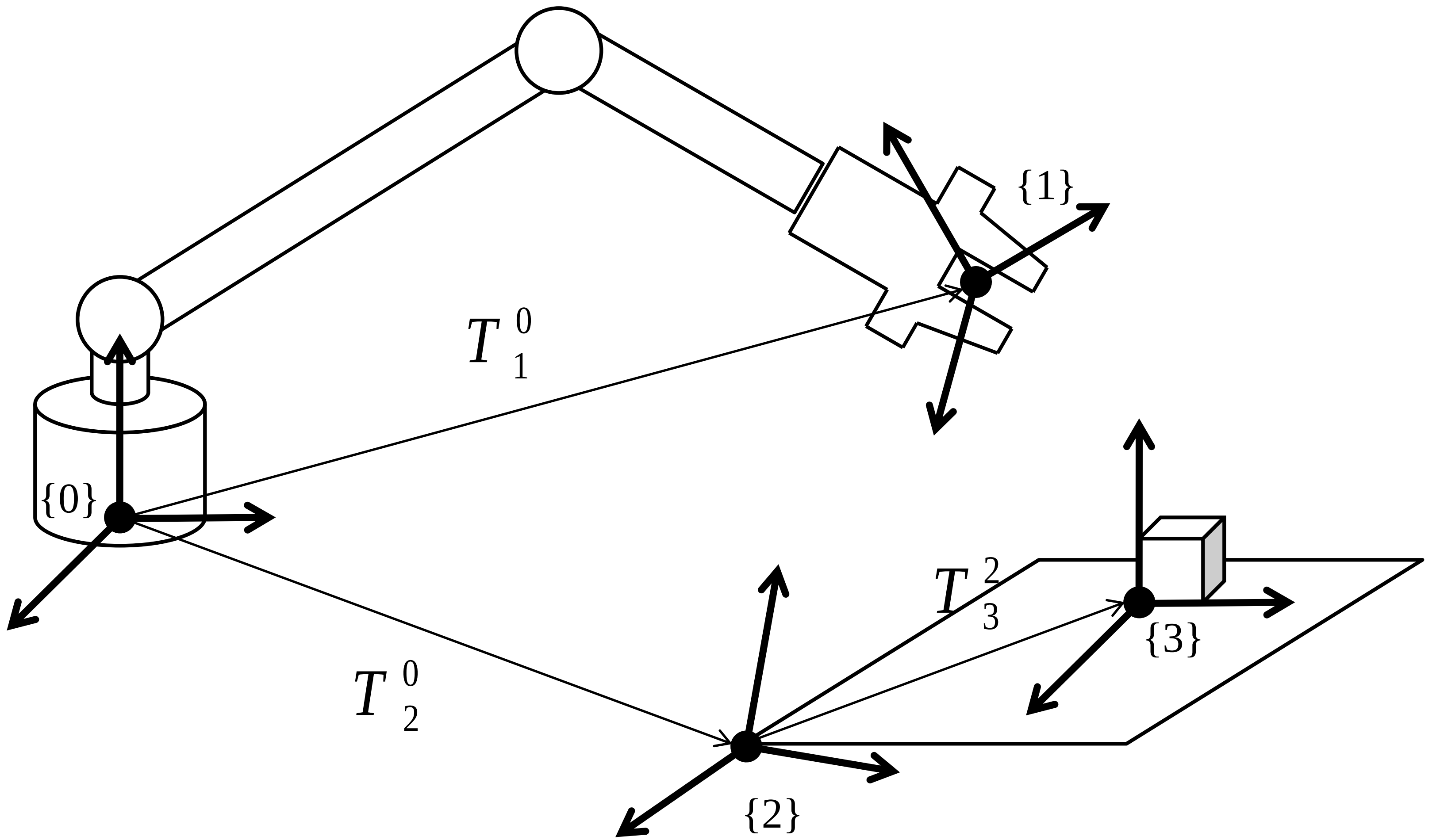
Transform Equations

- give expressions for:

$$T_2^0$$

$$T_4^3$$

Transform Equations



Transform Equations

- how can you find

$$T_1^0$$

$$T_2^0$$

$$T_3^2$$

$$T_3^1$$